# Section 1.2

eg $\{a_i\} = \left\{\frac{4^i}{i!}\right\}$

Here $a_{i+1} = \dfrac{4^{i+1}}{(i+1)!}$ so

$$\frac{a_{i+1}}{a_i} = \frac{4^{i+1}}{(i+1)!} \cdot \frac{i!}{4^i}$$

$$= \frac{4}{i+1}$$

Observe that $\frac{4}{i+1} < 1$ if $4 < i+1$ so $i > 3$.
However, because this does not hold for $i \leq 3$, $\{a_i\}$ is <u>not</u> monotonic.

<u>Def'n</u>: Given a sequence $\{a_i\}$ which is monotonic only for all $i \geq N$ where $N$ is a natural number, we say that $\{a_i\}$ has a <u>monotonic tail</u>.

eg $\{a_i\} = \left\{\frac{4^i}{i!}\right\}$ is decreasing for all $i \geq 4$, so it has a monotonic tail

<u>Def'n</u>: Given a sequence $\{a_i\}$, if there exists a real number $M$ such that $a_i \leq M$ for all $i$ then $\{a_i\}$ is <u>bounded above</u> and we call $M$ an <u>upper bound</u>. If there exists a real number $m$ such that $a_i \geq m$ for all $i$ then $\{a_i\}$ is <u>bounded below</u> and we call $m$ a <u>lower bound</u>. A sequence which is bounded above and bounded below is called a <u>bounded sequence</u>.

In general, we must rely on our understanding of the expressions which appear in the formula for $a_i$ to determine whether the sequence is bounded above, bounded below, both or neither.

However, if $\{a_i\}$ has been shown to be monotonic increasing then it must be bounded below and $a_1$ must be a lower bound. Likewise, if $\{a_i\}$ is monotonic decreasing then it must be bounded above with $a_1$ as an upper bound.

eg $\{a_i\} = \left\{\frac{i^2}{i^2+1}\right\}$ must be bounded below because we have shown that it is increasing.

Thus $a_1 = 1/2$ is a lower bound.

Furthermore because $i^2 < i^2+1$, $a_i < 1$ so $\{a_i\}$ is also bounded above with upper bound 1.

Therefore $\{a_i\}$ is a bounded sequence.

eg $\{a_i\} = \left\{\frac{4^i}{i!}\right\}$ is bounded above because it has a decreasing tail. We can see by computing the terms prior to the monotonic tail that $\frac{32}{3}$ is an upper bound.

Furthermore, $4^i > 0$ and $i! > 0$ so $a_i > 0$.

Hence $\{a_i\}$ must be bounded below with lower bound 0. Thus $\{a_i\}$ is a bounded sequence.

# The Bounded Monotonic Sequence Theorem (BMST)

If a sequence $\{a_i\}$ is bounded and has a monotonic tail then $\{a_i\}$ is convergent.

eg $\{a_i\} = \left\{\frac{i^2}{i^2+1}\right\}$ is convergent by the BMST

$\{a_i\} = \left\{\frac{4^i}{i!}\right\}$ is convergent by the BMST

Note that a sequence may not be monotonic (or have a monotonic tail) but still be convergent.

eg $\{a_i\} = \left\{\frac{(-1)^i i^2}{i^3+5}\right\}$ is not monotonic, yet we used the Abs. Sequence Theorem to show that it converges

## Section 1.3: Series

Given a sequence $\{a_i\} = \{a_1, a_2, a_3, a_4, \ldots\}$ we can define the infinite series

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + a_4 + \cdots$$

eg If $\{a_i\} = \left\{\frac{1}{i^2}\right\} = \{1, 1/4, 1/9, 1/16, \ldots\}$ then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

Is it meaningful to discuss the idea of finding a sum or "total" of an infinite series?