Section 1.2

Except when the limit of the corresponding function can be assigned $\pm\infty$, we cannot use the Evaluation Theorem to show that a sequence diverges.

eg $\{a_i\} = \{\sin(i\pi)\}$

The corresponding is $f(x) = \sin(\pi x)$ which oscillates between ± 1 repeatedly as $x \to \infty$, and thus $\lim_{x\to\infty} f(x)$ does not exist.

However, $\{a_i\} = \{0, 0, 0, 0, \dots\} = \{0\}$

so $\lim_{i\to\infty} a_i = 0$ and $\{a_i\}$ is convergent.

The Squeeze Theorem

Given sequences $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ if there exists a natural number N such that $c_i \leq a_i \leq b_i$ for all $i \geq N$, and if

$$\lim_{i\to\infty} b_i = \lim_{i\to\infty} c_i = L$$

then $\lim_{i\to\infty} a_i = L$ as well.

eg $\{a_i\} = \left\{\frac{\cos(i)}{i^2}\right\}$

We have $-1 \leq \cos(i) \leq 1$

$$-\frac{1}{i^2} \leq \frac{\cos(i)}{i^2} \leq \frac{1}{i^2}$$

Now $\lim_{i\to\infty} \frac{1}{i^2} = 0$ and $\lim_{i\to\infty} \frac{-1}{i^2} = 0$ so, by the Squeeze Theorem, $\lim_{i\to\infty} a_i = 0$ as well.

The Absolute Sequence Theorem

Given a sequence $\{a_i\}$, if $\lim_{i\to\infty} |a_i| = 0$ then $\lim_{i\to\infty} a_i = 0$.

Proof: Observe that

$$-|a_i| \leq a_i \leq |a_i|$$

But we are given that $\lim_{i\to\infty} |a_i| = 0$ so $\lim_{i\to\infty} -|a_i| = 0$.

By the Squeeze Theorem, $\lim_{i\to\infty} a_i = 0$ as well.

eg $\{a_i\} = \left\{ \frac{(-1)^i i^2}{i^3+5} \right\}$

The absolute sequence is

$$|a_i| = \frac{i^2}{i^3+5}$$

$$\lim_{i\to\infty} |a_i| = \lim_{i\to\infty} \frac{i^2}{i^3+5} \cdot \frac{1/i^3}{1/i^3}$$

$$= \lim_{i\to\infty} \frac{1/i}{1+5/i^3} = \frac{0}{1} = 0$$

Hence, by the Abs. Sequence Theorem, $\lim_{i\to\infty} a_i = 0$.

eg $\{a_i\} = \left\{ \frac{(-1)^i (5i-1)}{i+1} \right\}$

The absolute sequence is

$$|a_i| = \frac{5i-1}{i+1} \quad \text{so} \quad \lim_{i\to\infty} |a_i| = 5$$

We <u>cannot</u> conclude that $\lim_{i\to\infty} a_i = 5$.

However, we can conclude that the terms of $\{a_i\}$ eventually oscillate between approximately ± 5.

Therefore, $\{a_i\}$ must <u>diverge</u>.

Two key characteristics of a sequence are monotonicity and boundedness.

Def'n: A sequence $\{a_i\}$ is said to be increasing if $a_{i+1} > a_i$ for all i. It is decreasing if $a_{i+1} < a_i$ for all i. A sequence that is increasing or decreasing is monotonic.

One way to show monotonicity is to consider the corresponding function $f(x)$ and demonstrate that either $f'(x) > 0$ (increasing) or $f'(x) < 0$ (decreasing) for all $x \geq 1$.

eg $\{a_i\} = \left\{\frac{i^2}{i^2+1}\right\}$

The corresponding function is $f(x) = \frac{x^2}{x^2+1}$ so

$$f'(x) = \frac{2x(x^2+1) - 2x(x^2)}{(x^2+1)^2}$$

$$= \frac{2x}{(x^2+1)^2} > 0 \text{ for all } x \geq 1$$

Therefore $f(x)$ is increasing and hence $\{a_i\}$ is increasing as well, and thus monotonic.

Another approach is to consider $\frac{a_{i+1}}{a_i}$. If $\frac{a_{i+1}}{a_i} > 1$ for all i then $\{a_i\}$ is increasing. If $\frac{a_{i+1}}{a_i} < 1$ for all i then $\{a_i\}$ is decreasing.

eg $\{a_i\} = \left\{ \frac{2\cdot 4\cdot 6 \cdots (2i)}{1\cdot 3\cdot 5 \cdots (2i-1)} \right\}$

$$\frac{a_{i+1}}{a_i} = \frac{\frac{2\cdot 4\cdot 6 \cdots (2i)(2i+2)}{1\cdot 3\cdot 5 \cdots (2i-1)(2i+1)}}{\frac{2\cdot 4\cdot 6 \cdots (2i)}{1\cdot 3\cdot 5 \cdots (2i-1)}}$$

$$= \frac{2\cdot 4\cdot 6 \cdots (2i)(2i+2)}{1\cdot 3\cdot 5 \cdots (2i-1)(2i+1)} \cdot \frac{1\cdot 3\cdot 5 \cdots (2i-1)}{2\cdot 4\cdot 6 \cdots (2i)}$$

$$= \frac{(2i+2)}{(2i+1)}$$

$$= \frac{(2i+1)+1}{2i+1}$$

$$= 1 + \frac{1}{2i+1} > 1 \quad \text{for all } i$$

Thus $\{a_i\}$ is (monotonic) increasing.