Section 1.2

Theorem: Given a p-sequence $\{a_i\} = \{\frac{1}{i^p}\}$,

① if $p > 0$ then $\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{1}{i^p} = 0$

② if $p = 0$ then $\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{1}{i^p} = 1$

③ if $p < 0$ then $\{a_i\}$ is divergent

eg $\{a_i\} = \{\frac{1}{i^4}\}$ then $\lim_{i\to\infty} a_i = 0$ because $p > 0$

$\{a_i\} = \{\frac{1}{i^{-2}}\}$ then $\{a_i\}$ diverges because $p < 0$

Theorem: Given a geometric sequence $\{a_i\} = \{r^{i-1}\}$,

① if $|r| < 1$ then $\lim_{i\to\infty} a_i = \lim_{i\to\infty} r^{i-1} = 0$

② if $r = 1$ then $\lim_{i\to\infty} a_i = \lim_{i\to\infty} r^{i-1} = 1$

③ if $r = -1$ or $|r| > 1$ then $\{a_i\}$ diverges

eg $\{a_i\} = \{(\frac{1}{2})^{i-1}\}$ then $\lim_{i\to\infty} a_i = 0$ because $r = \frac{1}{2}$ so $|r| < 1$

$\{a_i\} = \{2^{i-1}\}$ then $\{a_i\}$ is divergent because $r = 2$ so $|r| > 1$

These results still hold even if a finite number of terms is removed from the start of the sequence.

eg $\{a_i\} = \{(-\frac{2}{3})^{i+2}\} = \{-\frac{8}{27}, \frac{16}{81}, -\frac{32}{243}, \dots\}$

This is the same as $\{b_i\} = \{(-\frac{2}{3})^{i-1}\}$ except that it is missing the first 3 terms so, again

$\lim_{i\to\infty} a_i = 0$ because $|r| = \frac{2}{3} < 1$

The basic properties of limits of functions also apply to limits of sequences.

eg $\lim_{i\to\infty}(a_i+b_i) = \lim_{i\to\infty} a_i + \lim_{i\to\infty} b_i$

if $\{a_i\}$ and $\{b_i\}$ are convergent sequences

eg $\lim_{i\to\infty} \frac{3i^2-2i+1}{4i^2+2} \cdot \frac{1/i^2}{1/i^2}$

$= \lim_{i\to\infty} \frac{3 - 2/i + 1/i^2}{4 + 2/i^2}$

$= \frac{3-0+0}{4+0} \boxed{= \frac{3}{4}}$ (converges)

eg $\lim_{i\to\infty} \frac{9^{i-1}}{2^{3i+1}} = \lim_{i\to\infty} \frac{9^i \cdot 9^{-1}}{2^{3i}\cdot 2^1}$

$= \lim_{i\to\infty} \frac{1}{18} \cdot \frac{9^i}{8^i}$

$= \frac{1}{18} \lim_{i\to\infty} \left(\frac{9}{8}\right)^i$

This diverges because $|r| = \frac{9}{8} > 1$.

eg $\lim_{i\to\infty} \frac{3^i+4^i}{6^i} = \lim_{i\to\infty}\left(\frac{3^i}{6^i} + \frac{4^i}{6^i}\right)$

$= \lim_{i\to\infty}\left[\left(\frac{1}{2}\right)^i + \left(\frac{2}{3}\right)^i\right]$

$= \lim_{i\to\infty}\left(\frac{1}{2}\right)^i + \lim_{i\to\infty}\left(\frac{2}{3}\right)^i$

$= 0 + 0$

$\boxed{= 0}$ (convergent)

eg $\lim_{i\to\infty} \frac{3^i}{4^i+6^i} \cdot \frac{1/6^i}{1/6^i}$

$= \lim_{i\to\infty} \frac{(1/2)^i}{(2/3)^i + 1}$

$= \frac{0}{0+1}$ $\boxed{=0}$ (convergent)

Some sequences may not be written as a combination of common sequences, but may be rewritten in terms of them.

eg $\lim_{i\to\infty} \frac{\cos(i\pi)}{2^i}$

$\cos(1\cdot\pi) = \cos(\pi) = -1$
$\cos(2\pi) = 1$
$\cos(3\pi) = -1$
$\cos(4\pi) = 1$ ...

$= \lim_{i\to\infty} \frac{(-1)^i}{2^i}$

$= \lim_{i\to\infty} \left(-\frac{1}{2}\right)^i$ $\boxed{=0}$ (convergent)

The Evaluation Theorem

Suppose that $\{a_i\}$ is a sequence and $f(x)$ is a function for which $f(i) = a_i$ for all i. If $\lim_{x\to\infty} f(x) = L$ then $\lim_{i\to\infty} a_i = L$ and so $\{a_i\}$ is convergent. If $\lim_{x\to\infty} f(x) = \pm\infty$ then $\{a_i\}$ diverges.

eg $\lim_{i\to\infty} \frac{\ln(i)+6i}{2i-5}$ Consider $f(x) = \frac{\ln(x)+6x}{2x-5}$

Since $\lim_{x\to\infty} f(x)$ is a $\frac{\infty}{\infty}$ form, we can apply l'Hôpital's Rule:

$$\lim_{x\to\infty} f(x) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}+6}{2} = \lim_{x\to\infty} \left(\frac{1}{2x} + 3\right) = 3$$

Thus, by the Evaluation Theorem,

$$\lim_{i\to\infty} \frac{\ln(i)+6i}{2i-5} = 3 \quad \text{(converges)}.$$

Section 2.1

A <u>function of three variables</u> $w = f(x,y,z)$ associates with each ordered triple (x,y,z) a unique real number w.

In general, a <u>function of n variables</u> $y = f(x_1, x_2, x_3, \ldots, x_n)$ associates with each ordered n-tuple $(x_1, x_2, x_3, \ldots, x_n)$ a unique real number y.

eg $f(x,y,z) = 3x^2 + yz$

$f(1,2,-3) = 3 - 6 = -3$ $\qquad$ $f(2,4,2) = 12 + 8 = 20$

$f(0,-5,1) = -5$ $\qquad$ $f(-1,-1,-1) = 3 + 1 = 4$

Section 2.2: Limits and Continuity

~~The~~ <u>Def'n</u>: Let L be a real number and (p,q) be a point. If $f(x,y)$ becomes arbitrarily close to L as (x,y) approaches (p,q) along every possible path then the <u>limit</u> of $f(x,y)$ as (x,y) tends towards (p,q) is L and we write

$$\lim_{(x,y)\to(p,q)} f(x,y) = L.$$

<u>Theorem</u>: If $f(x,y)$ is a polynomial function or a rational function for which $f(p,q)$ is defined then

$$\lim_{(x,y)\to(p,q)} f(x,y) = f(p,q)$$

eg $\displaystyle\lim_{(x,y)\to(2,-3)} (x^3 - 4xy^2 + 5y - 7) = 2^3 - 4.2(-3)^2 + 5(-3) - 7$

$\boxed{= -86}$

eg $\lim_{(x,y)\to(2,-2)} \dfrac{x^2-y^2}{x+y}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{(x,y)\to(2,-2)} \dfrac{(x-y)(x+y)}{x+y}$

$= \lim_{(x,y)\to(2,-2)} (x-y) = 2-(-2)$

$\boxed{=4}$

eg $\lim_{(x,y)\to(0,0)} \dfrac{3x+y}{x-y}$ $\left(\frac{0}{0} \text{ form}\right)$

We will try to show that this limit does <u>not</u> exist by finding two curves, each passing through $(0,0)$, along which $\dfrac{3x+y}{x-y}$ behaves differently.

First we try the horizontal line $y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{3x+y}{x-y} = \lim_{x\to 0} \frac{3x+0}{x-0}$$

$$= \lim_{x\to 0} 3$$

$$= 3$$

Next we try the vertical line $x=0$:

$$\lim_{(x,y)\to(0,0)} \frac{3x+y}{x-y} = \lim_{y\to 0} \frac{0+y}{0-y}$$

$$= \lim_{y\to 0} (-1)$$

$$= -1$$

Since these are different, we conclude that $\lim_{(x,y)\to(0,0)} \dfrac{3x+y}{x-y}$ does not exist.

eg $\lim_{(x,y)\to(1,0)} \frac{xy - y}{x^2 - 2x + y^2 + 1}$ ($\frac{0}{0}$ form)

Along $y = 0$, this becomes

$$\lim_{x\to 1} \frac{x\cdot 0 - 0}{x^2 - 2x + 0 + 1} = \lim_{x\to 1} 0 = 0$$

Along $x = 1$, the limit becomes

$$\lim_{y\to 0} \frac{1\cdot y - y}{1 - 2 + y^2 + 1} = \lim_{y\to 0} 0 = 0$$

Along $y = x - 1$, we have

$$\lim_{x\to 1} \frac{x(x-1) - (x-1)}{x^2 - 2x + (x-1)^2 + 1}$$

$$= \lim_{x\to 1} \frac{x^2 - 2x + 1}{2x^2 - 4x + 2}$$

$$= \frac{1}{2} \lim_{x\to 1} \frac{x^2 - 2x + 1}{x^2 - 2x + 1}$$

$$= \frac{1}{2} \lim_{x\to 1} 1$$

$$= \frac{1}{2}$$

Thus the limit does <u>not</u> exist.

<u>Def'n</u>: A function $f(x,y)$ is <u>continuous at a point</u> (p,q) if each of the following hold:

① $f(p,q)$ is defined

② $\lim_{(x,y)\to(p,q)} f(x,y)$ exists

③ $f(p,q) = \lim_{(x,y)\to(p,q)} f(x,y)$

If $f(x,y)$ is continuous at every ordered pair (p,q) then it is <u>everywhere continuous</u>.