Section 2.4

Theorem: The General Chain Rule

Suppose that z is a differentiable function of the n variables $u_1, u_2, u_3, \ldots, u_n$. Furthermore, suppose that each u_j is a differentiable function of the m variables $x_1, x_2, x_3, \ldots, x_m$. Then z is also a differentiable function of each x_i and

$$\frac{\partial z}{\partial x_i} = \frac{\partial z}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_i} + \frac{\partial z}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial z}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_i}$$

for any i such that $1 \le i \le m$.

eg Consider $z = u^2 v e^{-w}$ where $u = \arctan(xy^3)$, $v = \ln(x)$ and $w = 3x + 4y$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

We have

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= 2uve^{-w}\cdot\frac{1}{1+x^2y^6}\cdot y^3 + u^2e^{-w}\cdot\frac{1}{x} + (-u^2ve^{-w})\cdot 3$$

$$= \frac{2uve^{-w}y^3}{1+x^2y^6} + \frac{u^2e^{-w}}{x} - 3u^2ve^{-w}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$= 2uve^{-w}\cdot\frac{1}{1+x^2y^6}\cdot 3xy^2 + (-u^2ve^{-w})\cdot 4$$

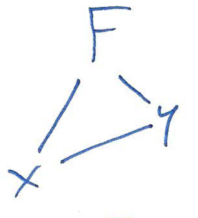
$$= \frac{6uve^{-w}xy^2}{1+x^2y^6} - 4u^2ve^{-w}$$

Suppose we have an equation which defines y implicitly as a function of x. This equation can be written in the form

$$F(x,y) = 0.$$

eg $x^3+y^4 = 6xy$

$x^3+y^4-6xy=0$ so $F(x,y) = x^3+y^4-6xy$



Since we can consider F as a function of x only, by the General Chain Rule,

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$

$$F_y\cdot\frac{dy}{dx} = -F_x \longrightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$$

eg (cont.)

$F(x,y) = x^3+y^4-6xy$

$F_x = 3x^2-6y$

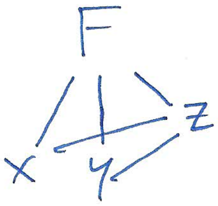
$F_y = 4y^3-6x$

so $$\frac{dy}{dx} = -\frac{3x^2-6y}{4y^3-6x} = \frac{6y-3x^2}{4y^3-6x}$$

Now consider an equation which defines z implicitly as a function of both x and y. This can be written in the form

$$F(x,y,z) = 0.$$

We wish to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. By the General Chain Rule,



$$\frac{\partial F}{\partial x} = F_x \cdot \frac{dx}{dx} + F_z \cdot \frac{\partial z}{\partial x}$$

$$0 = F_x + F_z \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Likewise,

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

eg $x^2y^3 + yz^4 = x - z$ Find $\frac{\partial z}{\partial x}$.

$$F(x,y,z) = x^2y^3 + yz^4 - x + z$$

$$F_x = 2xy^3 - 1$$

$$F_z = 4yz^3 + 1$$

Thus $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy^3-1}{4yz^3+1}$

$$= \frac{1-2xy^3}{4yz^3+1}$$

Section 2.5: Extreme Values

Def'n: A function $f(x,y)$ has a relative (or local) maximum at (p,q) if $f(x,y) \leq f(p,q)$ for all points (x,y) in an open circle of (p,q). If $f(x,y) \geq f(p,q)$ for all (x,y) in the open circle then $f(x,y)$ has a relative (or local) minimum at (p,q).

Def'n: A critical point of $f(x,y)$ is any point (p,q) at which both $f_x(p,q)=0$ and $f_y(p,q)=0$, or at which at least one of $f_x(p,q)$ or $f_y(p,q)$ is undefined.

eg $f(x,y) = x^2 + y^2$

$f_x(x,y) = 2x \qquad f_y(x,y) = 2y$

We set $2x = 0 \rightarrow x = 0$

and $2y = 0 \rightarrow y = 0$

Thus $(0,0)$ is the only critical point.

Any relative extremum must be a critical point, but not all critical points are relative extrema. When a critical point is not a relative extremum, it is called a saddle point.