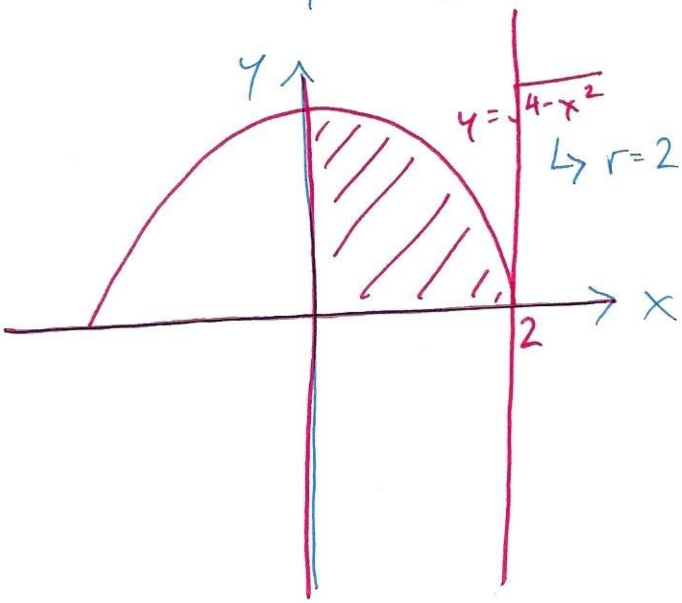
Section 2.9

eg $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2} e^{-y^2} \, dy \, dx$

The region of integration is bounded by the curves $y=0$, $y=\sqrt{4-x^2}$, $x=0$, $x=2$.



Observe that we can rewrite $y=\sqrt{4-x^2}$ as

$$y^2 = 4-x^2$$

$$x^2+y^2=4$$

Hence $y=\sqrt{4-x^2}$ is the upper semi-circle, centred at the origin, of radius 2.

In polar coordinates, this region is defined by

$$0 \le r \le 2, \quad 0 \le \theta \le \pi/2$$

Also, $e^{-x^2}e^{-y^2} = e^{-r^2\cos^2(\theta)} e^{-r^2\sin^2(\theta)}$

$$= e^{-r^2\cos^2(\theta) - r^2\sin^2(\theta)}$$

$$= e^{-r^2[\cos^2(\theta)+\sin^2(\theta)]}$$

$$= e^{-r^2}$$

Now we can rewrite the given integral as

$$\int_0^{\pi/2}\int_0^2 e^{-r^2}\cdot r\,dr\,d\theta$$

We let $u=-r^2$ so $-\frac{1}{2}du = r\,dr$

$r=0 \to u=0$

$r=2 \to u=-4$

Thus we have

$$-\frac{1}{2}\int_0^{\pi/2}\int_0^{-4} e^u\,du\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[e^u\right]_{u=0}^{u=-4} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}(e^{-4}-1)\,d\theta$$

$$= -\frac{1}{2}(e^{-4}-1)\left[\theta\right]_0^{\pi/2}$$

$$= -\frac{1}{2}(e^{-4}-1)\cdot\frac{\pi}{2} \boxed{= \frac{(1-e^{-4})\pi}{4}}$$

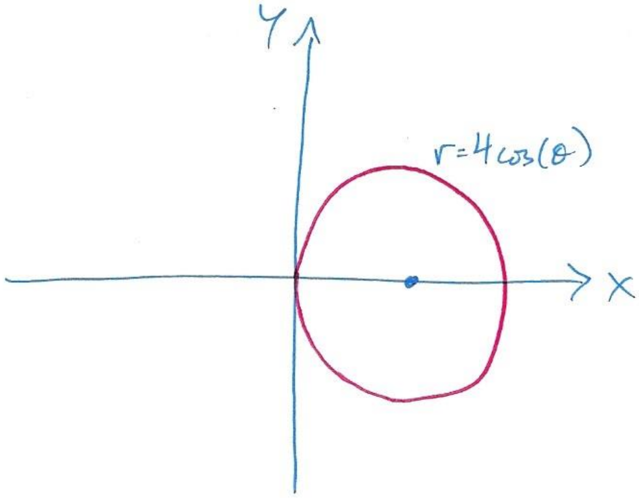
If any boundary curves are functions of r or functions of θ then these must be included in the inside integral when setting up an iterated integral in polar coordinates.

eg Find the area of the circle $(x-2)^2+y^2=4$.

Observe that $A = \iint_D dA = \iint_D r\,dr\,d\theta$.

In polar coordinates, the circle can be written

$$(x-2)^2+y^2=4$$
$$x^2-4x+4+y^2=4$$
$$x^2-4x+y^2=0$$
$$r^2\cos^2(\theta)-4r\cos(\theta)+r^2\sin^2(\theta)=0$$
$$r^2\left[\cos^2(\theta)+\sin^2(\theta)\right]-4r\cos(\theta)=0$$
$$r^2-4r\cos(\theta)=0$$
$$r=4\cos(\theta)$$



The circle D is defined by

$$0 \le r \le 4\cos(\theta)$$
$$-\pi/2 \le \theta \le \pi/2$$

Hence
$$A = \int_{-\pi/2}^{\pi/2}\int_0^{4\cos(\theta)} r\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4\cos(\theta)} d\theta$$
$$= 8\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\,d\theta$$
$$= 8\int_{-\pi/2}^{\pi/2}\frac{1+\cos(2\theta)}{2}\,d\theta$$
$$= 4\left[\theta+\tfrac{1}{2}\sin(2\theta)\right]_{-\pi/2}^{\pi/2}$$
$$= 4\left(\tfrac{\pi}{2}+\tfrac{\pi}{2}\right)$$
$$\boxed{=4\pi}$$

THE END