Section 1.11

A function whose domain is the set $\mathbb{C}$, or a subset of it, we call it <u>complex-valued</u>.

eg $f(x) = x^2$ where x is any complex number

$f(5) = 25$

$f(3i) = (3i)^2 = -9$

$f(2+i) = (2+i)^2 = 4+4i-1 = 3+4i$

But how do we define transcendental functions with complex domains?

Consider the function e^x. How would we define e^{ix} (for a real number x).

Recall that the Maclaurin series for e^x is

$$e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Then $$e^{ix} = \sum_{k=0}^{\infty} \frac{1}{k!} (ix)^k = \sum_{k=0}^{\infty} \frac{i^k}{k!} x^k.$$

Observe that

$i^0 = 1$ $\qquad$ $i^4 = 1$

$i^1 = i$ $\qquad$ $i^5 = i$

$i^2 = -1$ $\qquad$ $i^6 = -1$

$i^3 = -i$ $\qquad$ $i^7 = -i$ $\qquad$ and so on.

Now we write

$$e^{ix} = \sum_{\substack{k=0 \\ (k \text{ even})}}^{\infty} \frac{i^k}{k!} x^k + \sum_{\substack{k=0 \\ (k \text{ odd})}}^{\infty} \frac{i^k}{k!} x^k$$

$$= \sum_{\substack{k=0 \\ (k=2n)}}^{\infty} \frac{i^k}{k!} x^k + \sum_{\substack{k=0 \\ (k=2n+1)}}^{\infty} \frac{i^k}{k!} x^k$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= \cos(x) + i \sin(x)$$

This is <u>Euler's formula</u>.

Furthermore,

$$e^{\rho + i\theta} = e^{\rho} \cdot e^{i\theta} = e^{\rho}\left[\cos(\theta) + i\sin(\theta)\right]$$

$$= r\left[\cos(\theta) + i\sin(\theta)\right]$$

$$= r\cos(\theta) + ir\sin(\theta)$$

where $r = e^{\rho}$.

In order to write a complex number $\alpha + i\beta$ in an exponential form, we would need

$$\alpha = r\cos(\theta)$$
$$\beta = r\sin(\theta).$$

Then we must also have

$$r = \sqrt{\alpha^2+\beta^2} \quad \text{and} \quad \theta = \arctan\left(\frac{\beta}{\alpha}\right)$$

as a result of which $\alpha + i\beta = re^{i\theta}$.

This is the <u>polar form</u> of the complex number, and a complex number is often expressed in terms of its polar coordinates (r, θ).

eg Find the complex number z with polar coordinates $\left(2, \frac{3\pi}{4}\right)$.

If $r=2$ and $\theta = \frac{3\pi}{4}$ then

$$\alpha = 2\cos\left(\frac{3\pi}{4}\right) = 2\cdot\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

$$\beta = 2\sin\left(\frac{3\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2} = \sqrt{2}$$

So $z = -\sqrt{2} + i\sqrt{2}$.

eg Find the polar coordinates of $z = 3-3i$.

Since $\alpha = 3$ and $\beta = -3$,

$$r = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \arctan\left(\frac{-3}{3}\right) = \arctan(-1) = -\pi/4$$

so $z = (3\sqrt{2}, -\pi/4)$.

eg Compute $(3-3i)^6$.

We know that if $z = 3-3i$ then $r = 3\sqrt{2}$

and $\theta = -\pi/4$ so

$$z = 3\sqrt{2}\, e^{-\frac{\pi}{4}i}$$

$$z^6 = \left(3\sqrt{2}\, e^{-\frac{\pi}{4}i}\right)^6$$

$$= (3\sqrt{2})^6 e^{-\frac{6\pi}{4}i}$$

$$= 5832\, e^{-\frac{3\pi}{2}i}$$

$$= 5832\left[\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right]$$

$$= 5832\left[0 + i\cdot 1\right]$$

$$\boxed{= 5832i}$$

Also note that, in general,

$$(e^{i\theta})^n = e^{in\theta}$$

which is <u>de Moivre's Theorem</u>.