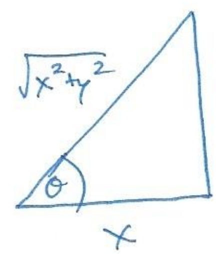
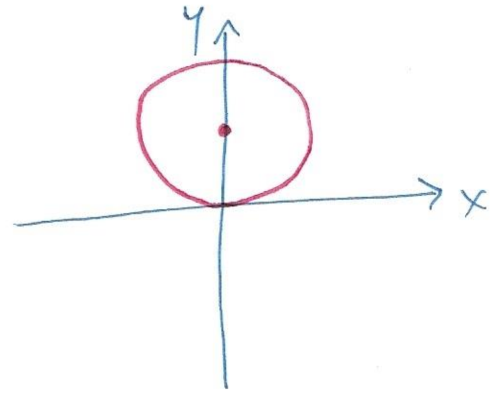
Section 2.8

Circles not centred at the origin do not have the form $r = k$.

eg $r = 7\sin(\theta)$



Converting back to Cartesian coordinates we have



$$\sqrt{x^2+y^2} = 7 \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 7y$$

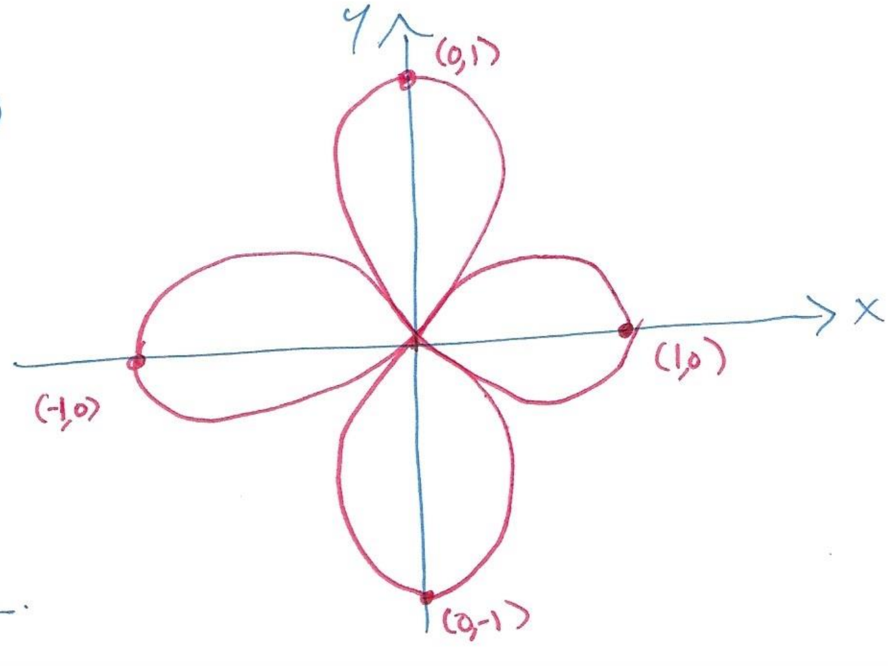
$$x^2+y^2-7y = 0$$

$$x^2+\left(y-\frac{7}{2}\right)^2 = \frac{49}{4}$$

This is a circle centred at $(0, 7/2)$ of radius $7/2$.

Some polar functions are best graphed directly in polar coordinates.

eg $r = \cos(2\theta)$



This is a 4-leaved rose.

Section 2.9: Double Integrals in Polar Coordinates

In Cartesian coordinates, a double integral over a region D has the form

$$\iint_D f(x,y)\, dA$$

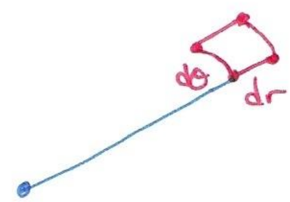
where $dA = dx\,dy$ or $dA = dy\,dx$.

Likewise, in polar coordinates it has the form

$$\iint_D f(r,\theta)\, dA$$

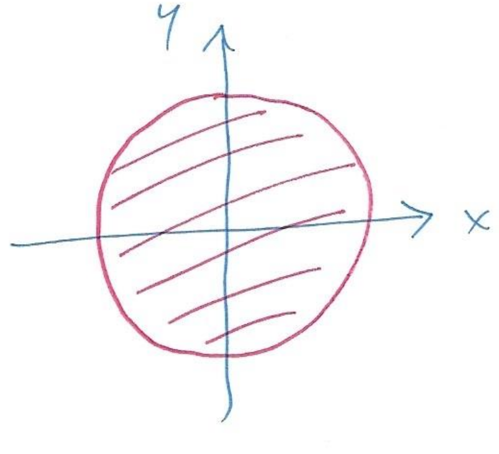
Here, however, we subdivide D into a number of <u>polar rectangles</u> with area

$$dA = r\,dr\,d\theta.$$



eg Find the volume of the hemisphere $z = \sqrt{1-x^2-y^2}$.

Recall that the region of integration D is the unit circle $x^2+y^2=1$, which is equivalent to $r=1$.



Here $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

Also, $\sqrt{1-x^2-y^2} = \sqrt{1-r^2\cos^2(\theta) - r^2\sin^2(\theta)}$

$= \sqrt{1-r^2\left[\cos^2(\theta)+\sin^2(\theta)\right]}$

$= \sqrt{1-r^2}$

So now $V = \iint_D \sqrt{1-x^2-y^2}\, dA$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1-r^2} \cdot r\, dr\, d\theta$$

We let $u = 1-r^2$ so $-\frac{1}{2} du = r\,dr$

$r = 0 \to u = 1$

$r = 1 \to u = 0$

Thus we have

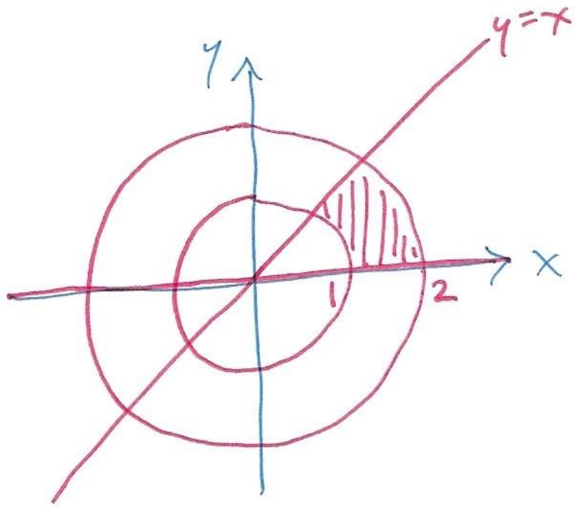
$$V = -\frac{1}{2} \int_0^{2\pi} \int_1^0 \sqrt{u}\, du\, d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left[\frac{2}{3} u^{3/2}\right]_{u=1}^{u=0} d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left(-\frac{2}{3}\right) d\theta$$

$$= \frac{1}{3} \left[\theta\right]_0^{2\pi} \boxed{= \frac{2\pi}{3}}$$

eg $\iint_D (3x+y)\, dA$ where D is the region in the first quadrant inside the circle $x^2+y^2=4$ outside the circle $x^2+y^2=1$ between the x-axis and the line $y=x$.



In polar coordinates,

$x^2+y^2=4 \longrightarrow r=2$

$x^2+y^2=1 \longrightarrow r=1$

$y=0 \longrightarrow \theta=0$

$y=x \longrightarrow \theta=\pi/4$

Next, $3x + y = 3r\cos(\theta) + r\sin(\theta)$

Hence $\iint_D (3x+y)\,dA = \int_0^{\pi/4} \int_1^2 [3r\cos(\theta) + r\sin(\theta)]\, r\,dr\,d\theta$

$$= \int_0^{\pi/4} \int_1^2 [3r^2\cos(\theta) + r^2\sin(\theta)]\, dr\,d\theta$$

$$= \int_0^{\pi/4} \int_1^2 r^2 [3\cos(\theta) + \sin(\theta)]\, dr\,d\theta$$

$$= \int_0^{\pi/4} \left[\frac{1}{3} r^3 [3\cos(\theta) + \sin(\theta)]\right]_{r=1}^{r=2} d\theta$$

$$= \frac{7}{3} \int_0^{\pi/4} [3\cos(\theta) + \sin(\theta)]\, d\theta$$

$$= \frac{7}{3} \left[3\sin(\theta) - \cos(\theta)\right]_0^{\pi/4}$$

$$= \frac{7}{3} \cdot (\sqrt{2} + 1) \boxed{= \frac{7\sqrt{2} + 7}{3}}$$