Section 1.10

eg Use a power series representation to find a series that represents $\int_0^1 e^{-x^2}\,dx$.

We have already shown that

$$e^{-x^2} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^{2i}$$

so

$$\int e^{-x^2}\,dx = C + \sum_{i=0}^{\infty}\left[\frac{(-1)^i}{i!}\int x^{2i}\,dx\right]$$

$$= C + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\cdot\frac{x^{2i+1}}{2i+1}$$

$$= C + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)} x^{2i+1}$$

Thus $\sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)} x^{2i+1}$ represents an antiderivative of e^{-x^2} and so, by the Fundamental Theorem of Calculus,

$$\int_0^1 e^{-x^2}\,dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)}\cdot 1^{2i+1} - \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)}\cdot 0^{2i+1}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)} - 0$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)}$$

We could approximate the definite integral with the first few terms of this alternating series, so

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx 0.7429.$$

Section 1.11: Complex Numbers

The natural numbers $\mathbb{N}$ consists of the whole numbers $1, 2, 3, 4, \ldots$.

The integers $\mathbb{Z}$ consist of $0, \pm 1, \pm 2, \pm 3, \ldots$

The rational numbers $\mathbb{Q}$ are quotients of the form $\frac{a}{b}$ where a, b are integers and $b \neq 0$.

The irrational numbers include any number whose decimal representation neither terminates nor repeats.

The real numbers $\mathbb{R}$ are comprised of all rational and irrational numbers.

However, there is no real number which is the solution to an equation such as $x^2 + 1 = 0$.

Such a solution would require $x^2 = -1$

$$x = \pm \sqrt{-1}$$

Thus we define an <u>imaginary</u> number i for which $i^2 = -1$.

From this, we define the <u>complex numbers</u> $\mathbb{C}$ to be any number of the form $z = \alpha + i\beta$ where α and β are real numbers. We call α the <u>real part</u> of z and β the <u>imaginary part</u>.

eg Solve $x^2+4x+5=0$

By the quadratic formula,

$$x = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm \sqrt{-1}\cdot\sqrt{4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

We can add or subtract complex numbers by adding or subtracting the real parts and their imaginary parts.

eg $(1+2i)+(3+4i) = (1+3)+(2+4)i$

$= 4+6i$

We multiply complex numbers using distributivity:

eg $(1+2i)(3+4i) = 1\cdot3 + 1\cdot4i + 2i\cdot3 + 2i\cdot4i$

$= 3+4i+6i+8i^2$

$= 3+10i-8$

$= -5+10i$

Two complex numbers $z=\alpha+i\beta$ and $w=\gamma+i\delta$ are <u>equal</u> if $\alpha=\gamma$ and $\beta=\delta$.

Observe that if $\alpha + i\beta = 0$ then $\alpha = \beta = 0$ because

$$0 + i0 = 0.$$

Some polynomial expressions can only be factored in terms of complex numbers.

$$\text{eg} \quad x^2 + 9 = (x - 3i)(x + 3i)$$

In general, $x^2 + k^2 = (x - ki)(x + ki)$.

Consider $z = \alpha + i\beta$. Then

$$\frac{1}{z} = \frac{1}{\alpha + i\beta} \cdot \frac{\alpha - i\beta}{\alpha - i\beta}$$

$$= \frac{\alpha - i\beta}{\alpha^2 - i\alpha\beta + i\alpha\beta - i^2\beta^2}$$

$$= \frac{\alpha - i\beta}{\alpha^2 + \beta^2}$$

$$= \frac{\alpha}{\alpha^2 + \beta^2} - i\frac{\beta}{\alpha^2 + \beta^2}$$

We call $\alpha - i\beta$ the <u>complex conjugate</u> of $\alpha + i\beta$.

Given a complex number z, we denote its conjugate by $\bar{z}$.

$$\text{eg} \quad \frac{2+2i}{2-3i} = \frac{2+2i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{4+6i+4i-6}{4+6i-6i+9}$$

$$= \frac{-2+10i}{13} = -\frac{2}{13} + \frac{10}{13}i$$

Given $z = \alpha + i\beta$ then

$$z \cdot \bar{z} = (\alpha + i\beta)(\alpha - i\beta)$$
$$= \alpha^2 + \beta^2$$

which is a real number.

We define the absolute value of <u>modulus</u> of z to be

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{\alpha^2 + \beta^2}.$$