MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 8	**MATH 2000**	Fall 2018

SOLUTIONS

[4] 1. (a) We use the Ratio Test applied to the coefficients, with

$$k_i = \frac{1 \cdot 3 \cdot 5 \cdots (2i+1)}{(2i)!} \quad \text{and} \quad k_{i+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2i+3)}{(2i+2)!}.$$

Then

$$\begin{aligned}
\rho &= \lim_{i\to\infty} \left| \frac{k_{i+1}}{k_i} \right| \\
&= \lim_{i\to\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2i+3)}{(2i+2)!} \cdot \frac{(2i)!}{1 \cdot 3 \cdot 5 \cdots (2i+1)} \\
&= \lim_{i\to\infty} \frac{2i+3}{(2i+1)(2i+2)} \\
&= \lim_{i\to\infty} \frac{2i+3}{4i^2+6i+2} \\
&= 0
\end{aligned}$$

so the radius of convergence is $R = \infty$. Hence the power series converges for all real numbers $\mathbb{R}$.

[6] (b) First we have to rewrite the power series as

$$\sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{6^i \ln(i)} (3-2x)^i = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{6^i \ln(i)} \cdot (-2)^i \left(x - \frac{3}{2}\right)^i = \sum_{i=0}^{\infty} \frac{-1}{3^i \ln(i)} \left(x - \frac{3}{2}\right)^i.$$

Now we can apply the Ratio Test with

$$k_i = \frac{-1}{3^i \ln(i)} \quad \text{and} \quad k_{i+1} = \frac{-1}{3^{i+1} \ln(i+1)}.$$

Then

$$\begin{aligned}
\rho &= \lim_{i\to\infty} \left| \frac{k_{i+1}}{k_i} \right| \\
&= \lim_{i\to\infty} \frac{1}{3^{i+1} \ln(i+1)} \cdot 3^i \ln(i) \\
&= \lim_{i\to\infty} \frac{\ln(i)}{3 \ln(i+1)} \\
&= \frac{1}{3}
\end{aligned}$$

by the Evaluation Theorem and l'Hôpital's Rule. Hence the radius of convergence is $R = \frac{1}{\rho} = 3$ and so the power series certainly converges for $\left|x - \frac{3}{2}\right| < 3$, that is, for $-3 < x - \frac{3}{2} < 3$ or $-\frac{3}{2} < x < \frac{9}{2}$.
At $x = \frac{9}{2}$, the series becomes

$$\sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{6^i \ln(i)}(-6)^i = \sum_{i=0}^{\infty} \frac{-1}{\ln(i)},$$

which diverges by the Limit Comparison Test with test series $\sum \frac{1}{i}$.
At $x = -\frac{3}{2}$, the series becomes

$$\sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{6^i \ln(i)} 6^i = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{\ln(i)},$$

which converges by the Alternating Series Test.
Thus the interval of convergence is $-\frac{3}{2} \leq x < \frac{9}{2}$.

[6] (c) Because of the power of $3i$, we need to apply the Root Test to the entire expression of the terms of the series, and not just the coefficients. We have

$$\begin{aligned} L &= \lim_{i\to\infty} |a_i|^{\frac{1}{i}} \\ &= \lim_{i\to\infty} \frac{\left(i^{\frac{1}{i}}\right)^4}{8^{2-\frac{1}{i}}} |x|^3 \\ &= \frac{|x|^3}{64}. \end{aligned}$$

Thus the series will certainly converge when

$$\frac{|x|^3}{64} < 1 \quad \Longrightarrow \quad -64 < x^3 < 64 \quad \Longrightarrow \quad -4 < x < 4.$$

Hence the radius of convergence is $R = 4$.
At $x = 4$, the series becomes

$$\sum_{i=0}^{\infty} (-1)^i \frac{i^4}{8^{2i-1}} 4^{3i} = \sum_{i=0}^{\infty} (-1)^i \frac{8i^4}{64^i} 64^i = 8 \sum_{i=0}^{\infty} (-1)^i i^4,$$

which diverges by the Divergence Test.
Similarly, at $x = -4$, the series becomes

$$\sum_{i=0}^{\infty} (-1)^i \frac{i^4}{8^{2i-1}} (-4)^{3i} = \sum_{i=0}^{\infty} \frac{8i^4}{64^i} 64^i = 8 \sum_{i=0}^{\infty} i^4,$$

which also diverges by the Divergence Test.
Thus the interval of convergence is $-4 < x < 4$.

[4] 2. (a) First we set

$$\frac{1}{16}x^2 = \frac{1}{2}\sqrt{x} \quad \Longrightarrow \quad x^2 = 8\sqrt{x} \quad \Longrightarrow \quad x^4 = 64x \quad \Longrightarrow \quad x(x^3 - 64) = 0$$

so the intersection points of the two curves are $x = 0$ and $x = 4$. On this region, $y = \frac{1}{2}\sqrt{x}$ is the upper boundary curve and $y = \frac{1}{16}x^2$ is the lower boundary curve, so

$$\begin{aligned} V &= \int_0^4 \int_{\frac{1}{16}x^2}^{\frac{1}{2}\sqrt{x}} xy^2 \, dy \, dx \\ &= \int_0^4 \left[\frac{1}{3}xy^3\right]_{y=\frac{1}{16}x^2}^{\frac{1}{2}\sqrt{x}} dx \\ &= \int_0^4 \left[\frac{1}{24}x^{\frac{5}{2}} - \frac{1}{12288}x^7\right] dx \\ &= \left[\frac{1}{84}x^{\frac{7}{2}} - \frac{1}{98304}x^8\right]_0^4 \\ &= \frac{32}{21} - \frac{2}{3} \\ &= \frac{6}{7}. \end{aligned}$$

[4] (b) As functions of y, the two boundary curves become $x = 4\sqrt{y}$ and $x = 4y^2$. Using the information we found in part (a), when $x = 0$ then $y = 0$, and when $x = 4$ then $y = 1$. Since $x = 4\sqrt{y}$ is the right boundary curve and $x = 4y^2$ is the left boundary curve, we have

$$\begin{aligned} V &= \int_0^1 \int_{4y^2}^{4\sqrt{y}} xy^2 \, dx \, dy \\ &= \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{x=4y^2}^{x=4\sqrt{y}} dy \\ &= \int_0^1 \left[8y^3 - 8y^6\right] dy \\ &= \left[2y^4 - \frac{8}{7}y^7\right]_0^1 \\ &= 2 - \frac{8}{7} \\ &= \frac{6}{7}. \end{aligned}$$

[5] 3. The points $(-1, 4)$ and $(1, 2)$ are joined by the line $y = 3 - x$. The points $(1, 2)$ and $(2, 4)$ are joined by the line $y = 2x$. The points $(-1, 4)$ and $(2, 4)$ are joined by the line $y = 4$.

This is a Type II region, so we can rewrite the left boundary curve $y = 3 - x$ as $x = 3 - y$, and we can rewrite the right boundary curve $y = 2x$ as $x = \frac{1}{2}y$. Then

$$\begin{aligned} A &= \int_2^4 \int_{3-y}^{\frac{1}{2}y} dx\, dy \\ &= \int_2^4 \Big[y \Big]_{x=3-y}^{x=\frac{1}{2}y} dy \\ &= \int_2^4 \left[\frac{1}{2}y - (3 - y) \right] dy \\ &= \int_2^4 \left(\frac{3}{2}y - 3 \right) dy \\ &= \left[\frac{3}{4}y^2 - 3y \right]_2^4 \\ &= 12 - 12 - 3 + 6 \\ &= 3. \end{aligned}$$

[5] 4. This is both a Type I and Type II region, but we cannot integrate e^{2x^3} with respect to x. Thus we must start with an integral with respect to y, which means that we will treat this as a Type I region, bounded above by $y = x^2$ and below by $y = 0$ (which intersect at $x = 0$). Hence

$$\begin{aligned} \iint_D e^{2x^3}\, dA &= \int_0^1 \int_0^{x^2} e^{2x^3}\, dy\, dx \\ &= \int_0^1 \Big[ye^{2x^3} \Big]_{y=0}^{y=x^2} dx \\ &= \int_0^1 x^2 e^{2x^3}\, dx. \end{aligned}$$

Now we let $u = 2x^3$ so $\frac{1}{6}\, du = x^2\, dx$. When $x = 0$, $u = 0$. When $x = 1$, $u = 2$. The integral becomes

$$\begin{aligned} \iint_D e^{2x^3}\, dA &= \frac{1}{6} \int_0^2 e^u\, du \\ &= \frac{1}{6} \Big[e^u \Big]_0^2 \\ &= \frac{1}{6}(e^2 - 1). \end{aligned}$$

[6] 5. As given, this is a Type I region bounded above by $y = \sqrt{\pi}$ and below by $y = x^3$ on the interval from $x = 0$ to $x = \sqrt[6]{\pi}$. However, we cannot find an antiderivative of $\sin(y^2)$. So instead we will interpret this as a Type II region bounded to the right by $x = y^{\frac{1}{3}}$ and to the left by $x = 0$ on the interval from $y = 0$ to $y = \sqrt{\pi}$. Then

$$\begin{aligned}\int_0^{\sqrt[6]{\pi}} \int_{x^3}^{\sqrt{\pi}} x^2 \sin(y^2)\, dy\, dx &= \int_0^{\sqrt{\pi}} \int_0^{y^{\frac{1}{3}}} x^2 \sin(y^2)\, dx\, dy \\ &= \int_0^{\sqrt{\pi}} \left[\frac{1}{3} x^3 \sin(y^2)\right]_{x=0}^{x=y^{\frac{1}{3}}} dy \\ &= \frac{1}{3} \int_0^{\sqrt{\pi}} y \sin(y^2)\, dy.\end{aligned}$$

Let $u = y^2$ so $\frac{1}{2}\, du = y\, dy$. When $y = 0$, $u = 0$. When $y = \sqrt{\pi}$, $u = \pi$. The integral becomes

$$\begin{aligned}\int_0^{\sqrt[6]{\pi}} \int_{x^3}^{\sqrt{\pi}} x^2 \sin(y^2)\, dy\, dx &= \frac{1}{6} \int_0^{\pi} \sin(u)\, du \\ &= \frac{1}{6} \Big[-\cos(u)\Big]_0^{\pi} \\ &= \frac{1}{6}(1+1) \\ &= \frac{1}{3}.\end{aligned}$$