MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 | **MATH 2000** | FALL 2018

SOLUTIONS

[3] 1. (a) First consider the absolute series $\sum_{i=1}^{\infty} \frac{2i+1}{3i^3-2}$. Note that, for large i, this resembles the series $\sum_{i=1}^{\infty} \frac{i}{i^3} = \sum_{i=1}^{\infty} \frac{1}{i^2}$, which is a convergent p-series. Then

$$\lim_{i\to\infty} \frac{\frac{2i+1}{3i^3-2}}{\frac{1}{i^2}} = \lim_{i\to\infty} \frac{2i^3+i}{3i^3-2} = \frac{2}{3},$$

and so, by the Limit Comparison Test, $\sum_{i=1}^{\infty} \frac{2i+1}{3i^3-2}$ also converges. We conclude that the given series is <u>absolutely convergent</u>.

[5] (b) First consider the absolute series $\sum_{i=1}^{\infty} \frac{2i^2+i}{3i^3-2}$. Note that, for large i, this resembles $\sum_{i=1}^{\infty} \frac{i^2}{i^3} = \sum_{i=1}^{\infty} \frac{1}{i}$, which is the (divergent) harmonic series. Then

$$\lim_{i\to\infty} \frac{\frac{2i^2+i}{3i^3-2}}{\frac{1}{i}} = \lim_{i\to\infty} \frac{2i^3+i^2}{3i^3-2} = \frac{2}{3},$$

and so, by the Limit Comparison Test, $\sum_{i=1}^{\infty} \frac{2i^2+i}{3i^3-2}$ also diverges. We conclude that the given series is not absolutely convergent.

So now we return to the original series $\sum_{i=1}^{\infty} (-1)^i \frac{2i^2+i}{3i^3-2}$. Let $p_i = \frac{2i^2+i}{3i^3-2}$. Immediately we can see that $\lim_{i\to\infty} p_i = 0$. From $f(x) = \frac{2x^2+x}{3x^3-2}$, we have

$$f'(x) = -\frac{2(3x^4+3x^3+4x+1)}{(3x^3-2)^2}.$$

Since $f'(x) < 0$ for $x \geq 1$, $\{p_i\}$ is decreasing for $i \geq 1$, and therefore fulfills the requirements of the Alternating Series Test. Thus $\sum_{i=1}^{\infty} (-1)^i \frac{2i^2+i}{3i^3-2}$ is convergent, and consequently it is <u>conditionally convergent</u>.

[2] (c) Observe that

$$\lim_{i\to\infty} \frac{2i^3+i}{3i^3-2} = \frac{2}{3},$$

so the terms of the series alternately approach $\pm\frac{2}{3}$. Hence, by the Divergence Test, this series is <u>divergent</u>.

[4] (d) We use the Ratio Test, with

$$|a_i| = \frac{7^i}{1\cdot 3\cdot 5\cdots(2i+1)} \quad \text{so} \quad |a_{i+1}| = \frac{7^{i+1}}{1\cdot 3\cdot 5\cdots(2i+3)}.$$

Thus

$$L = \lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty} \frac{7^{i+1}}{1\cdot 3\cdot 5\cdots(2i+3)} \cdot \frac{1\cdot 3\cdot 5\cdots(2i+1)}{7^i} = \lim_{i\to\infty}\frac{7}{2i+3} = 0.$$

Since $L<1$, we conclude that the given series is <u>absolutely convergent</u>.

[4] (e) We use the Ratio Test, with

$$|a_i| = \frac{(3i)!}{(i!)^3} \quad \text{so} \quad |a_{i+1}| = \frac{(3i+3)!}{[(i+1)!]^3}.$$

Thus

$$L = \lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty}\frac{(3i+3)!}{[(i+1)!]^3}\cdot\frac{(i!)^3}{(3i)!} = \lim_{i\to\infty}\frac{(3i+1)(3i+2)(3i+3)}{(i+1)^3} = 27.$$

Since $L>1$, the given series must be <u>divergent</u>.

[4] (f) We use the Root Test, with

$$|a_i| = \frac{i}{[\arctan(i)]^i} \quad \text{so} \quad |a_i|^{\frac{1}{i}} = \frac{i^{\frac{1}{i}}}{\arctan(i)}.$$

Then

$$L = \lim_{i\to\infty}|a_i|^{\frac{1}{i}} = \lim_{i\to\infty}\frac{i^{\frac{1}{i}}}{\arctan(i)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

Since $L<1$, we conclude that the given series is <u>absolutely convergent</u>.

[4] 2. Let $p_i = \dfrac{1}{4i^3}$. It's clear that $\lim_{i\to\infty} p_i = 0$ and since $4i^3 < 4(i+1)^3$, it must be that $p_i < p_{i+1}$ so $\{p_i\}$ is decreasing. Hence this series meets the requirements of the Alternating Series Test. Then

$$|R_n| = |a_{n+1}| = \frac{1}{4(n+1)^3}.$$

To achieve the desired accuracy, we set

$$\begin{aligned}
\frac{1}{4(n+1)^3} &= 0.001 = \frac{1}{1000}\\
4(n+1)^3 &= 1000\\
n+1 &= \sqrt[3]{250} \approx 6.3\\
n &\approx 5.3.
\end{aligned}$$

Hence the partial sum s_6 will guarantee accuracy to within 0.0001, and

$$s_6 = \frac{1}{4} - \frac{1}{32} + \frac{1}{108} - \frac{1}{256} + \frac{1}{500} - \frac{1}{864} = \frac{194353}{864000} \approx 0.2249.$$

(In fact, the true sum of the series is about 0.225386.)

[3] 3. (a) Let $u = x^2 y$ so $\frac{1}{2}\,du = xy\,dx$. The integral becomes

$$\begin{aligned} \int xy\cos(x^2y)\,dx &= \frac{1}{2}\int \cos(u)\,du \\ &= \frac{1}{2}\sin(u) + C(y) \\ &= \frac{1}{2}\sin(x^2y) + C(y). \end{aligned}$$

[3] (b) We use integration by parts with $w = xy$ so $dw = x\,dy$, and $dv = \cos(x^2y)\,dy$ so $v = \frac{1}{x^2}\sin(x^2y)$. Thus

$$\begin{aligned} \int xy\cos(x^2y)\,dy &= xy\cdot\frac{1}{x^2}\sin(x^2y) - \int \frac{1}{x^2}\sin(x^2y)\cdot x\,dy \\ &= \frac{y}{x}\sin(x^2y) - \int \frac{1}{x}\sin(x^2y)\,dy \\ &= \frac{y}{x}\sin(x^2y) - \frac{1}{x}\cdot\left[-\frac{1}{x^2}\cos(x^2y)\right] + C(x) \\ &= \frac{y}{x}\sin(x^2y) + \frac{1}{x^3}\cos(x^2y) + C(x). \end{aligned}$$

[4] 4. First observe that the function which describes the plane is

$$z = 4 - 2x + \frac{1}{2}y.$$

Thus we must compute

$$\iint\limits_R \left(4 - 2x + \frac{1}{2}y\right)\,dA.$$

It might be slightly easier to begin by integrating with respect to y (since $y = 0$ is one of the bounds), although it won't matter much either way:

$$\begin{aligned}
\iint_R \left(4 - 2x + \frac{1}{2}y\right) dA &= \int_{-2}^{2}\int_{0}^{5} \left(4 - 2x + \frac{1}{2}y\right) dy\,dx \\
&= \int_{-2}^{2} \left[4y - 2xy + \frac{1}{4}y^2\right]_{y=0}^{y=5} dx \\
&= \int_{-2}^{2} \left[\left(20 - 10x + \frac{25}{4}\right) - 0\right] dx \\
&= \int_{-2}^{2} \left(\frac{105}{4} - 10x\right) dx \\
&= \left[\frac{105}{4}x - 5x^2\right]_{-2}^{2} \\
&= \left(\frac{105}{2} - 50\right) - \left(-\frac{105}{2} - 50\right) \\
&= 105.
\end{aligned}$$

[4] 5. This time, in the absence of any other considerations, the $x = 0$ bound suggests that we begin by integrating with respect to x:

$$\begin{aligned}
\iint_R (9 - x^2 + y^2)\, dA &= \int_{-1}^{2}\int_{0}^{3} (9 - x^2 + y^2)\, dx\,dy \\
&= \int_{-1}^{2} \left[9x - \frac{1}{3}x^3 + xy^2\right]_{x=0}^{x=3} dy \\
&= \int_{-1}^{2} [(27 - 9 + 3y^2) - 0]\, dy \\
&= \int_{-1}^{2} (18 + 3y^2)\, dy \\
&= \left[18y + y^3\right]_{-1}^{2} \\
&= (36 + 8) - (-18 - 1) \\
&= 63.
\end{aligned}$$