MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4 | **MATH 2000** | FALL 2018

SOLUTIONS

[3] 1. (a) We merely note that

$$\lim_{i\to\infty} \frac{i^3-3}{3i^3-i} = \frac{1}{3} \neq 0,$$

so by the Divergence Test, this is a divergent series.

[8] (b) We can perform a partial fraction decomposition and write

$$\frac{i-3}{i^3-i} = \frac{i-3}{i(i-1)(i+1)} = -\frac{1}{i-1} + \frac{3}{i} - \frac{2}{i+1}.$$

Recalling that the first term corresponds to $i = 2$, this means that the first few terms of the sequence of partial sums are

$$\begin{aligned}
s_1 &= -\frac{1}{1} + \frac{3}{2} - \frac{2}{3} \\
s_2 &= \left(-\frac{1}{1} + \frac{3}{2} - \frac{2}{3}\right) + \left(-\frac{1}{2} + \frac{3}{3} - \frac{2}{4}\right) \\
s_3 &= \left(-\frac{1}{1} + \frac{3}{2} - \frac{2}{3}\right) + \left(-\frac{1}{2} + \frac{3}{3} - \frac{2}{4}\right) + \left(-\frac{1}{3} + \frac{3}{4} - \frac{2}{5}\right) \\
s_4 &= \left(-\frac{1}{1} + \frac{3}{2} - \frac{2}{3}\right) + \left(-\frac{1}{2} + \frac{3}{3} - \frac{2}{4}\right) + \left(-\frac{1}{3} + \frac{3}{4} - \frac{2}{5}\right) + \left(-\frac{1}{4} + \frac{3}{5} - \frac{2}{6}\right).
\end{aligned}$$

Observe that the $-\frac{2}{3}$ in the first term, the $+\frac{3}{3}$ in the second term, and the $-\frac{1}{3}$ in the third term all cancel out. Similarly, the $-\frac{2}{4}$ in the second term, the $+\frac{3}{4}$ in the third term, and the $-\frac{1}{4}$ in the fourth term all cancel out. Thus, for example, we can simplify s_4 above to become

$$s_4 = -\frac{1}{1} + \frac{3}{2} - \frac{1}{2} - \frac{2}{5} + \frac{3}{4} - \frac{2}{6}.$$

The telescoping cancels out everything except the first two numbers in the first term, the first number in the second term, the last number in the second-last term, and the last two numbers in the last term.

Following this pattern, then, we can see that

$$\begin{aligned}
s_n &= -\frac{1}{1} + \frac{3}{2} - \frac{1}{2} - \frac{2}{n+1} + \frac{3}{n+1} - \frac{2}{n+2} \\
&= \frac{1}{n+1} - \frac{2}{n+2} \\
\lim_{n\to\infty} s_n &= 0 - 0 \\
&= 0.
\end{aligned}$$

Thus we conclude that this is a convergent series, and

$$\sum_{i=1}^{\infty} \frac{i^3-3}{3i^3-i} = 0.$$

[5] (c) We will first attempt to use the Divergence Test. We consider the corresponding function $f(x) = \sqrt[x]{x} = x^{\frac{1}{x}}$. As $x \to \infty$ this is an ∞^0 indeterminate form, so we use l'Hôpital's Rule, letting

$$y = \ln\left(x^{\frac{1}{x}}\right) = \frac{\ln(x)}{x}.$$

Then

$$\lim_{x\to\infty} y \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0,$$

and so

$$\lim_{x\to\infty} f(x) = e^0 = 1.$$

By the Evaluation Theorem, we can conclude that

$$\lim_{i\to\infty} \sqrt[i]{i} = 1 \neq 0$$

as well, and thus this series is divergent by the Divergence Test.

[3] (d) Observe that

$$\lim_{i\to\infty} \ln\left(\frac{2i-1}{3i+1}\right) = \ln\left(\lim_{i\to\infty} \frac{2i-1}{3i+1}\right) = \ln\left(\frac{2}{3}\right) \neq 0.$$

Hence the series is divergent by the Divergence Test.

[6] (e) We can write

$$\ln\left(\frac{2i-1}{2i+1}\right) = \ln(2i-1) - \ln(2i+1).$$

Thus the first few terms of the sequence of partial sums are

$$\begin{aligned}
s_1 &= \ln(1) - \ln(3) \\
s_2 &= [\ln(1) - \ln(3)] + [\ln(3) - \ln(5)] = \ln(1) - \ln(5) \\
s_3 &= [\ln(1) - \ln(3)] + [\ln(3) - \ln(5)] + [\ln(5) - \ln(7)] = \ln(1) - \ln(7) \\
s_4 &= [\ln(1) - \ln(3)] + [\ln(3) - \ln(5)] + [\ln(5) - \ln(7)] + [\ln(7) - \ln(9)] = \ln(1) - \ln(9).
\end{aligned}$$

In general, we can see that

$$\begin{aligned}
s_n &= \ln(1) - \ln(2n+1) \\
&= -\ln(2n+1) \\
\lim_{n\to\infty} s_n &= -\infty.
\end{aligned}$$

Since the sequence of partial sums diverges, we can conclude that this series is also divergent by the Divergent Test.

[4] 2. By the Chain Rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

Here,

$$\frac{\partial z}{\partial x} = \cos(x)\cos(y), \quad \frac{\partial z}{\partial y} = -\sin(x)\sin(y), \quad \frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = \frac{1}{\sqrt{t}}.$$

Hence

$$\frac{dz}{dt} = \frac{\cos(x)\cos(y)}{t} - \frac{\sin(x)\sin(y)}{2\sqrt{t}}.$$

[6] 3. By the Chain Rule,

$$f_x = f_\alpha \alpha_x + f_\beta \beta_x, \quad \text{and} \quad f_y = f_\alpha \alpha_y + f_\beta \beta_y$$

but

$$f_z = f_\beta \beta_z$$

because α does not depend on z. We have

$$f_\alpha = \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}}, \quad f_\beta = \frac{-\beta}{\sqrt{\alpha^2 - \beta^2}}$$

$$\alpha_x = \tan(y), \quad \alpha_y = x\sec^2(y)$$

$$\beta_x = \frac{3x^2}{y+3z}, \quad \beta_y = \frac{-x^3}{(y+3z)^2}, \quad \beta_z = \frac{-3x^3}{(y+3z)^2}.$$

Thus

$$f_x = \frac{\alpha\tan(y)}{\sqrt{\alpha^2 - \beta^2}} - \frac{3\beta x^2}{(y+3z)\sqrt{\alpha^2 - \beta^2}}$$

$$f_y = \frac{\alpha x\sec^2(y)}{\sqrt{\alpha^2 - \beta^2}} + \frac{\beta x^3}{(y+3z)\sqrt{\alpha^2 - \beta^2}}$$

$$f_z = \frac{3\beta x^3}{(y+3z)\sqrt{\alpha^2 - \beta^2}}.$$

[5] 4. We rewrite the equation as $x^2y + y^3z^2 - z^4x^3 - xe^y\cosh(z) = 0$ and set

$$F(x, y, z) = x^2y + y^3z^2 - z^4x^3 - xe^y\cosh(z).$$

Then

$$F_x = 2xy - 3z^4x^2 - e^y\cosh(z)$$

$$F_y = x^2 + 3y^2z^2 - xe^y\cosh(z)$$

$$F_z = 2y^3z - 4z^3x^3 - xe^y\sinh(z).$$

Hence we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy - 3z^4x^2 - e^y\cosh(z)}{2y^3z - 4z^3x^3 - xe^y\sinh(z)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2 + 3y^2z^2 - xe^y\cosh(z)}{2y^3z - 4z^3x^3 - xe^y\sinh(z)}.$$