MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 | **MATH 2000** | FALL 2018

SOLUTIONS

[3] 1. (a) We can write

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{(4\cdot 1)\cdot(4\cdot 2)\cdot(4\cdot 3)\cdots(4\cdot i)}{4^i} = \lim_{i\to\infty} \frac{4^i i!}{4^i} = \lim_{i\to\infty} i! = \infty.$$

Hence $\{a_i\}$ is divergent.

[4] (b) Observe that

$$\begin{aligned}
\lim_{i\to\infty} |a_i| &= \lim_{i\to\infty} \frac{9-4^{i+2}}{4^{2i-1}+3^i} \\
&= \lim_{i\to\infty} \frac{9-4^i\cdot 4^2}{4^{2i}\cdot 4^{-1}+3^i} \\
&= \lim_{i\to\infty} \frac{9-16\cdot 4^i}{\frac{1}{4}\cdot 16^i+3^i}\cdot\frac{\frac{1}{16^i}}{\frac{1}{16^i}} \\
&= \lim_{i\to\infty} \frac{\frac{9}{16^i}-16\left(\frac{1}{4}\right)^i}{\frac{1}{4}+\left(\frac{3}{16}\right)^i} \\
&= \frac{0-0}{\frac{1}{4}+0} \\
&= 0.
\end{aligned}$$

Thus, by the Absolute Sequence Theorem,

$$\lim_{i\to\infty} a_i = 0.$$

[4] (c) This time,

$$\begin{aligned}
\lim_{i\to\infty} |a_i| &= \lim_{i\to\infty} \frac{7^i-3^i}{7^i+3^i}\cdot\frac{\frac{1}{7^i}}{\frac{1}{7^i}} \\
&= \lim_{i\to\infty} \frac{1-\left(\frac{3}{7}\right)^i}{1+\left(\frac{3}{7}\right)^i} \\
&= \frac{1-0}{1+0} \\
&= 1.
\end{aligned}$$

Since $\lim_{i\to\infty} a_i \neq 0$, the Absolute Sequence Theorem does not apply. However, this suggests that the odd terms of $\{a_i\}$ are tending towards -1 while the even terms tend towards 1. Hence $\{a_i\}$ is a divergent sequence.

[4] (d) Since

$$-1 \leq \cos(3i) \leq 1,$$

we can write

$$-\frac{i^2}{4i^3 - 2i^2 + 5} \leq \frac{i^2 \cos(3i)}{4i^3 - 2i^2 + 5} \leq \frac{i^2}{4i^3 - 2i^2 + 5}.$$

However,

$$\lim_{i\to\infty} \frac{i^2}{4i^3 - 2i^2 + 5} \cdot \frac{\frac{1}{i^3}}{\frac{1}{i^3}} = \lim_{i\to\infty} \frac{\frac{1}{i}}{4 - \frac{2}{i^2} + \frac{5}{i^3}} = \frac{0}{4 - 0 + 0} = 0.$$

Similarly,

$$\lim_{i\to\infty} -\frac{i^2}{4i^3 - 2i^2 + 5} = 0.$$

Thus, by the Squeeze Theorem,

$$\lim_{i\to\infty} a_i = 0.$$

[5] (e) The corresponding function is

$$f(x) = [e^x + x]^{\frac{1}{x}}.$$

As $x \to \infty$, $f(x) \to \infty^0$, so this is an indeterminate form. Instead, we consider

$$\begin{aligned} y &= \ln\left([e^x + x]^{\frac{1}{x}}\right) \\ &= \frac{1}{x}\ln(e^x + x) \\ &= \frac{\ln(e^x + x)}{x}. \end{aligned}$$

As $x \to \infty$, $y \to \frac{\infty}{\infty}$, so l'Hôpital's Rule applies. We have

$$\begin{aligned} \lim_{x\to\infty} y &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x + x} \cdot (e^x + 1)}{1} \\ &= \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \\ &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \\ &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} \\ &= \lim_{x\to\infty} 1 \\ &= 1. \end{aligned}$$

Hence

$$\lim_{x\to\infty} f(x) = e^1 = e$$

and so, by the Evaluation Theorem,

$$\lim_{i\to\infty} a_i = e$$

as well.

[6] 2. (a) Consider the corresponding function $f(x) = \sqrt{x^2+8} - x$. We have

$$f'(x) = \frac{1}{2\sqrt{x^2+8}} \cdot 2x - 1 = \frac{x}{\sqrt{x^2+8}} - 1.$$

Observe that $\sqrt{x^2+8} > \sqrt{x^2} = x$, and so $\dfrac{x}{\sqrt{x^2+8}} < 1$. Now we can conclude that

$$f'(x) < 1 - 1 = 0,$$

and therefore $\{a_i\}$ is monotonic decreasing.
For the bounds, we can now immediately conclude that $a_1 = 2$ is an upper bound. Furthermore, by the same reasoning as above,

$$a_i = \sqrt{i^2+8} - i > i - i = 0,$$

so 0 is a lower bound. Hence $\{a_i\}$ is bounded, with $0 < a_i < 2$ for all i.

[6] (b) Here,

$$a_i = \frac{(i!)^2}{(2i)!} \quad \Longrightarrow \quad a_{i+1} = \frac{[(i+1)!]^2}{(2i+2)!}.$$

We compute

$$\frac{a_{i+1}}{a_i} = \frac{[(i+1)!]^2}{(2i+2)!} \cdot \frac{(2i)!}{(i!)^2} = \frac{(i+1)^2}{(2i+1)(2i+2)} = \frac{i+1}{2(2i+1)}.$$

Observe that $2i+1 > i+1$ so

$$\frac{a_{i+1}}{a_i} = \frac{i+1}{2(2i+1)} < \frac{1}{2} < 1.$$

Thus $\{a_i\}$ is monotonic decreasing.
This also tells us that $a_1 = \frac{1}{2}$ is an upper bound. Additionally, $(i!)^2 > 0$ and $(2i)! > 0$ so $a_i > 0$ for all i. Hence 0 is a lower bound. We can therefore conclude that $\{a_i\}$ is bounded, with $0 < a_i < \frac{1}{2}$ for all i.

[2] 3. (a) First we have

$$f_x(x,y) = 6x^2y^2 - 7y^3 + 9y - 6x$$
$$f_y(x,y) = 4x^3y - 21xy^2 + 9x - 1.$$

Then

$$f_{xx}(x,y) = 12xy^2 - 6$$
$$f_{xy}(x,y) = 12x^2y - 21y^2 + 9$$
$$f_{yy}(x,y) = 4x^3 - 42xy.$$

[2] (b) We have

$$f_r(r,\theta) = \frac{\theta^2}{1+r^2\theta^4}$$
$$f_\theta(r,\theta) = \frac{2r\theta}{1+r^2\theta^4}$$

and so

$$f_{rr}(r,\theta) = \frac{-2r\theta^6}{(1+r^2\theta^4)^2}$$
$$f_{r\theta}(r,\theta) = \frac{2\theta(1+r^2\theta^4) - 4r^2\theta^3(\theta^2)}{(1+r^2\theta^4)^2} = \frac{2\theta - 2r^2\theta^5}{(1+r^2\theta^4)^2}$$
$$f_{\theta\theta}(r,\theta) = \frac{2r(1+r^2\theta^4) - 4r^2\theta^3(\theta^2)}{(1+r^2\theta^4)^2} = \frac{2r + 2r^3\theta^4 - 4r^2\theta^5}{(1+r^2\theta^4)^2}.$$

[4] 4. We have

$$\frac{\partial z}{\partial x} = 3e^y - 7ye^x, \quad \frac{\partial z}{\partial y} = 3xe^y - 7e^x$$
$$\frac{\partial^2 z}{\partial x^2} = -7ye^x, \quad \frac{\partial^2 z}{\partial x \partial y} = 3e^y - 7e^x, \quad \frac{\partial^2 z}{\partial y^2} = 3xe^y$$
$$\frac{\partial^3 z}{\partial x^3} = -7ye^x, \quad \frac{\partial^3 z}{\partial x^2 \partial y} = -7e^x, \quad \frac{\partial^3 z}{\partial x \partial y^2} = 3e^y, \quad \frac{\partial^3 z}{\partial y^3} = 3xe^y.$$

Thus

$$\frac{\partial^3 z}{\partial x^3} - y\frac{\partial^3 z}{\partial x^2 \partial y} = -7ye^x - y(-7e^x) = -7ye^x + 7ye^x = 0$$

and

$$x\frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^3 z}{\partial y^3} = x(3e^y) - 3xe^y = 3xe^y - 3xe^y = 0$$

as well. Since the two sides of the PDE are equal, we can conclude that $z = 3xe^y - 7ye^x$ is a solution of the PDE.