MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.6 | **Math 2000 Worksheet** | FALL 2018

SOLUTIONS

1. (a) The absolute series is $\displaystyle\sum_{i=2}^{\infty} \frac{\ln(i)}{i}$. On Worksheet 1.4, we used the Integral Test to show that this series diverges. Otherwise, we could also deduce this by trying the Limit Comparison Test with the (divergent) harmonic series $\displaystyle\sum_{i=2}^{\infty} \frac{1}{i}$:

$$\lim_{i\to\infty} \frac{\frac{\ln(i)}{i}}{\frac{1}{i}} = \lim_{i\to\infty} \ln(i) = \infty.$$

Hence the given series is not absolutely convergent. To see if it's conditionally convergent, we use the Alternating Series Test. Observe that

$$\lim_{i\to\infty} \frac{\ln(i)}{i} \overset{\text{H}}{=} \lim_{i\to\infty} \frac{\frac{1}{i}}{1} = \lim_{i\to\infty} \frac{1}{i} = 0,$$

as required. Also, letting $f(x) = \frac{\ln(x)}{x}$, we have

$$f'(x) = \frac{1-\ln(x)}{x^2} < 0 \quad \text{for } x > 2,$$

so $\left\{\frac{\ln(i)}{i}\right\}$ is decreasing. Thus, by the Alternating Series Test, the given series is convergent. Since it is convergent but not absolutely convergent, we conclude that the given series is conditionally convergent.

(b) The absolute series is $\displaystyle\sum_{i=1}^{\infty} \frac{1}{3i^2+1}$. To test its convergence, we use the Limit Comparison Test with the convergent p-series $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2}$:

$$\lim_{i\to\infty} \frac{\frac{1}{3i^2+1}}{\frac{1}{i^2}} = \lim_{i\to\infty} \frac{i^2}{3i^2+1} = \frac{1}{3},$$

so the absolute series is convergent. Hence the given series must also converge, and it is absolutely convergent.

(c) Observe that

$$\lim_{i\to\infty} \frac{\sqrt{i}}{1+4\sqrt{i}} = \frac{1}{4},$$

so by the Divergence Test, the given series is divergent. (Note that this means its absolute series diverges also.)

(d) The absolute series is $\sum_{i=1}^{\infty} \frac{\sqrt[3]{i}}{i} = \sum_{i=1}^{\infty} \frac{1}{i^{\frac{2}{3}}}$, which is a divergent p-series. So the given series is not absolutely convergent. We use the Alternating Series Test on the given series. We observe that

$$\lim_{i \to \infty} \frac{1}{i^{\frac{2}{3}}} = 0.$$

Also, setting $f(x) = \frac{1}{x^{\frac{2}{3}}} = x^{-\frac{2}{3}}$, we have

$$f'(x) = -\frac{2}{3}x^{-\frac{5}{3}} < 0,$$

so $\left\{ \frac{1}{i^{\frac{2}{3}}} \right\}$ is decreasing. Hence, by the Alternating Series Test, the given series is convergent, and so it is conditionally convergent.

(e) The absolute series is $\sum_{i=1}^{\infty} \frac{1}{e^{i^3}}$. To determine its convergence, we use the Direct Comparison Test with the convergent geometric series $\sum_{i=1}^{\infty} \left(\frac{1}{e}\right)^i$, observing that for $i \geq 1$,

$$\begin{aligned} (e^i)^3 &> e^i \\ \frac{1}{(e^i)^3} &< \frac{1}{e^i} \\ \frac{1}{e^{i^3}} &< \left(\frac{1}{e}\right)^i. \end{aligned}$$

Hence the absolute series convergent, and therefore the given series is absolutely convergent.

(f) First note that $\cos(i\pi) = (-1)^i$ so the series can be written $\sum_{i=1}^{\infty} \frac{(-1)^i}{i^{\frac{1}{4}}}$ and the absolute series is $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{4}}}$. This is a divergent p-series, so the given series is not absolutely convergent. To test the convergence of the given series, we use the Alternating Series Test. Note that

$$\lim_{i \to \infty} \frac{1}{i^{\frac{1}{4}}} = 0.$$

Also, letting $f(x) = \frac{1}{x^{\frac{1}{4}}} = x^{-\frac{1}{4}}$, we have

$$f'(x) = -\frac{1}{4}x^{-\frac{5}{4}} < 0$$

so $\left\{ \frac{1}{i^{\frac{1}{4}}} \right\}$ is decreasing. Thus, by the Alternating Series Test, the given series is convergent and hence conditionally convergent.

2. Let $a_i = \frac{1}{4^i i!}$. First note that

$$\lim_{i\to\infty} \frac{1}{4^i i!} = 0.$$

To check that $\{a_i\}$ is decreasing, note that $a_{i+1} = \frac{1}{4^{i+1}(i+1)!}$ so

$$\frac{a_{i+1}}{a_i} = \frac{1}{4^{i+1}(i+1)!} \cdot \frac{4^i i!}{1} = \frac{4^i}{4^{i+1}} \cdot \frac{i!}{(i+1)!} = \frac{1}{4(i+1)} < 1 \quad \text{for } i \geq 1.$$

Hence we can use the remainder estimate for the Alternating Series Test. The sum of the first 5 terms of the series is

$$s_5 = -\frac{1}{4} + \frac{1}{32} - \frac{1}{384} + \frac{1}{6144} - \frac{1}{122880} \approx -0.2211995.$$

So we know that

$$\begin{array}{rcccl} & & |R_5| & < & |a_6| = \frac{1}{2949120} \approx 0.0000003 \\ -0.0000003 & < & s - s_5 & < & 0.0000003 \\ -0.2211998 & < & s & < & -0.2211992 \end{array}$$

where s is the sum of the series.

3. (a) We use the Ratio Test, letting

$$a_i = (-1)^i \frac{i^3}{3^i} \quad \text{so} \quad a_{i+1} = (-1)^{i+1} \frac{(i+1)^3}{3^{i+1}}.$$

Then

$$\lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i\to\infty} \frac{(i+1)^3}{3^{i+1}} \cdot \frac{3^i}{i^3} = \lim_{i\to\infty} \frac{i^3 + 3i^2 + 3i + 1}{3i^3} = \frac{1}{3} = L.$$

Since $L < 1$, the given series converges.

(b) We use the Ratio Test, letting

$$a_i = \frac{4^i}{i!} \quad \text{so} \quad a_{i+1} = \frac{4^{i+1}}{(i+1)!}.$$

Then

$$\lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i\to\infty} \frac{4^{i+1}}{(i+1)!} \cdot \frac{i!}{4^i} = \lim_{i\to\infty} \frac{4}{i+1} = 0 = L.$$

Since $L < 1$, the given series converges.

(c) We use the Root Test, letting $a_i = \left(\frac{3i}{i+2} \right)^i$. Then

$$\lim_{i\to\infty} |a_i|^{\frac{1}{i}} = \lim_{i\to\infty} \frac{3i}{i+2} = 3 = L.$$

Since $L > 1$, the given series diverges.

(d) We use the Root Test, letting $a_i = i\left(\frac{1}{7}\right)^{2i}$. Then

$$\lim_{i\to\infty} |a_i|^{\frac{1}{i}} = \lim_{i\to\infty} i^{\frac{1}{i}} \left(\frac{1}{7}\right)^2 = \frac{1}{49} \lim_{i\to\infty} i^{\frac{1}{i}} = \frac{1}{49} = L.$$

Since $L < 1$, the given series converges.

(e) We use the Ratio Test, letting

$$a_i = \frac{i^i}{i!} \quad \text{so} \quad a_{i+1} = \frac{(i+1)^{i+1}}{(i+1)!}.$$

Then

$$\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = \lim_{n\to\infty} \frac{(i+1)^{i+1}}{(i+1)!} \cdot \frac{i!}{i^i} = \lim_{i\to\infty} \frac{(i+1)^{i+1}}{(i+1)i^i} = \lim_{i\to\infty} \frac{(i+1)^i}{i^i} = \lim_{i\to\infty} \left(\frac{i+1}{i}\right)^i.$$

This is a 1^∞ indeterminate form, so we let $f(x) = \left(\frac{x+1}{x}\right)^x$ and set

$$y = \left(\frac{x+1}{x}\right)^x$$

$$\ln(y) = x \ln\left(\frac{x+1}{x}\right) = \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{x}{x+1}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x+1}{x} = 1$$

so

$$\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = e^1 = e = L,$$

and since $L > 1$, the series diverges.

(f) Let $a_i = \frac{1}{i^3\sqrt{i}} = \frac{1}{i^{\frac{7}{2}}}$. First we try the Ratio Test, for which $a_{i+1} = \frac{1}{(i+1)^{\frac{7}{2}}}$:

$$\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty} \frac{1}{(i+1)^{\frac{7}{2}}} \cdot i^{\frac{7}{2}} = \lim_{i\to\infty} \left(\frac{i}{i+1}\right)^{\frac{7}{2}} = 1^{\frac{7}{2}} = 1 = L,$$

so the Ratio Test fails. So instead we try the Root Test:

$$\lim_{i\to\infty} |a_i|^{\frac{1}{i}} = \lim_{i\to\infty} \frac{1}{i^{\frac{7}{2i}}} = \lim_{i\to\infty} \left(\frac{1}{i^{\frac{1}{i}}}\right)^{\frac{7}{2}} = 1^{\frac{7}{2}} = 1 = L,$$

so the Root Test also fails. (In fact, it can easily be seen that the given series is a convergent p-series.)

(g) We use the Root Test, letting $a_i = (-1)^{i+1}\left(\frac{3}{2} - \sqrt[i]{i}\right)^i$. Then

$$\lim_{i\to\infty} |a_i|^{\frac{1}{i}} = \lim_{i\to\infty}\left(\frac{3}{2} - \sqrt[i]{i}\right) = \lim_{i\to\infty}\left(\frac{3}{2} - i^{\frac{1}{i}}\right) = \frac{3}{2} - 1 = \frac{1}{2} = L.$$

So since $L < 1$, the given series converges.

(h) We use the Ratio Test, letting

$$a_i = \frac{2^i}{i^2+6} \quad \text{so} \quad a_{i+1} = \frac{2^{i+1}}{(i+1)^2+6}.$$

Then

$$\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty}\frac{2^{i+1}}{(i+1)^2+6}\cdot\frac{i^2+6}{2^i} = \lim_{i\to\infty}\frac{2(i^2+6)}{(i+1)^2+6} = 2 = L.$$

So since $L > 1$, the given series diverges.

(i) We use the Ratio Test, letting

$$a_i = \frac{(4i)!}{(i!)^3} \quad \text{so} \quad a_{i+1} = \frac{(4i+4)!}{((i+1)!)^3}.$$

Then

$$\begin{aligned}\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| &= \lim_{i\to\infty}\frac{(4i+4)!}{((i+1)!)^3}\cdot\frac{(i!)^3}{(4i)!}\\ &= \lim_{i\to\infty}\frac{(4i+1)(4i+2)(4i+3)(4i+4)}{(i+1)^3} = \infty = L.\end{aligned}$$

Since $L > 1$, the given series diverges.