MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 2.3 **Math 2000 Worksheet** Winter 2017

SOLUTIONS

1. (a) We construct the table:

i	$f^{(i)}(x)$	$f^{(i)}(0)$	k_i
0	xe^x	0	0
1	$e^x + xe^x$	1	$\frac{1}{1!} = \frac{1}{0!}$
2	$2e^x + xe^x$	2	$\frac{2}{2!} = \frac{1}{1!}$
3	$3e^x + xe^x$	3	$\frac{3}{3!} = \frac{1}{2!}$
4	$4e^x + xe^x$	4	$\frac{4}{4!} = \frac{1}{3!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$ie^x + xe^x$	i	$\frac{i}{i!} = \frac{1}{(i-1)!}$

So we see that

$$xe^x = 0 + \frac{1}{0!}x + \frac{1}{1!}x^2 + \frac{1}{2!}x^3 + \frac{1}{3!}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{1}{(i-1)!}x^i.$$

(b) We construct the table:

i	$f^{(i)}(x)$	$f^{(i)}(1)$	k_i
0	$x^{\frac{1}{2}}$	1	$\frac{1}{0!}$
1	$\frac{1}{2}x^{-\frac{1}{2}}$	$\frac{1}{2}$	$\frac{1}{2(1!)}$
2	$-\frac{1}{4}x^{-\frac{3}{2}}$	$-\frac{1}{4}$	$-\frac{1}{4(2!)}$
3	$\frac{3}{8}x^{-\frac{5}{2}}$	$\frac{3}{8}$	$\frac{3}{8(3!)}$
4	$-\frac{15}{16}x^{-\frac{7}{2}}$	$-\frac{15}{16}$	$-\frac{15}{16(4!)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$(-1)^i \frac{(-1)\cdot 1\cdot 3\cdot 5\cdots(2i-3)}{2^i} x^{-\frac{2i-1}{2}}$	$(-1)^{i+1} \frac{1\cdot 3\cdot 5\cdots(2i-3)}{2^i}$	$(-1)^{i+1} \frac{1\cdot 3\cdot 5\cdots(2i-3)}{2^i(i!)}$

So we see that

$$\begin{aligned}\sqrt{x} &= 1 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{3}{8}(x-1)^3 - \frac{15}{16}(x-1)^4 + \cdots \\ &= 1 + \sum_{i=1}^{\infty} (-1)^i \frac{(-1)\cdot 1\cdot 3\cdot 5\cdots(2i-3)}{2^i(i!)}(x-1)^i.\end{aligned}$$

(c) We construct the table:

i	$f^{(i)}(x)$	$f^{(i)}\left(\frac{\pi}{4}\right)$	k_i
0	$\sin(2x)$	1	$\frac{1}{0!}$
1	$2\cos(2x)$	0	0
2	$-4\sin(2x)$	-4	$\frac{-4}{2!}$
3	$-8\cos(2x)$	0	0
4	$16\sin(2x)$	16	$\frac{16}{4!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i (odd)	$(-1)^{\frac{i-1}{2}}2^i\cos(2x)$	0	0
i (even)	$(-1)^{\frac{i}{2}}2^i\sin(2x)$	$(-1)^{\frac{i}{2}}2^i$	$\frac{(-1)^{\frac{i}{2}}2^i}{i!}$

So we have that

$$\sin(2x) = \frac{1}{0!} - \frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{16}{4!}\left(x-\frac{\pi}{4}\right)^4 - \cdots = \sum_{i=0}^{\infty}\frac{(-1)^i 2^{2i}}{(2i)!}\left(x-\frac{\pi}{4}\right)^{2i}.$$

(d) We construct the table:

i	$f^{(i)}(x)$	$f^{(i)}(-2)$	k_i
0	x^{-4}	$\frac{1}{16}$	$\frac{1}{16(0!)}$
1	$-4x^{-5}$	$\frac{4}{32}$	$\frac{4}{32(1!)}$
2	$20x^{-6}$	$\frac{20}{64}$	$\frac{20}{64(2!)}$
3	$-120x^{-7}$	$\frac{120}{128}$	$\frac{120}{128(3!)}$
4	$840x^{-8}$	$\frac{840}{256}$	$\frac{840}{256(4!)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$(-1)^i[4\cdot5\cdot6\cdots(i+3)]x^{-(i+4)}$	$\frac{4\cdot5\cdot6\cdots(i+3)}{2^{i+4}}$	$\frac{4\cdot5\cdot6\cdots(i+3)}{2^{i+4}i!}$

So then

$$\begin{aligned} x^{-4} &= \frac{1}{16(0!)} + \frac{4}{32(1!)}(x+2) + \frac{20}{64(2!)}(x+2)^2 + \frac{120}{128(3!)}(x+2)^3 + \cdots \\ &= \frac{1}{16} + \sum_{i=1}^{\infty}\frac{4\cdot5\cdot6\cdots(i+3)}{2^{i+4}i!}(x+2)^i. \end{aligned}$$

2. Since $-1 \le x \le 1$, we have

$$f^{(3)}(x) = 3e^x + xe^x \le 3e + e = 4e \approx 10.87.$$

Furthermore, $|x-p| = |x| \le 1$. Thus

$$R_2(x) \le \frac{4e}{3!}\cdot 1 \approx 1.81.$$

Following the pattern which can be observed in the table we generated for Question 1(a), we can see that

$$f^{(11)}(x) = 11e^x + xe^x \le 11e + e = 12e \approx 32.62.$$

Now

$$R_{10}(x) \leq \frac{12e}{11!} \cdot 1 \approx 0.00000082.$$

3. (a) We have

$$e^x = \sum_{n=0}^{\infty} \frac{1}{i!} x^i$$

$$e^{-\frac{x}{4}} = \sum_{i=0}^{\infty} \frac{1}{i!} \left(-\frac{x}{4}\right)^i$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{4^i i!} x^i.$$

(b) We have

$$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1}$$

$$\sin(x^6) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} (x^6)^{2i+1}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{12i+6}.$$

(c) We have

$$\cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i}$$

$$x\cos(x) = x \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i+1}$$

(d) Observe first that

$$\ln\left(\frac{1-2x}{1+2x}\right) = \ln(1-2x) - \ln(1+2x).$$

Now we know that

$$\ln(1+x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^{i+1}$$

so

$$\ln(1-2x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1}(-2x)^{i+1} = \sum_{i=0}^{\infty} \frac{(-1)^i(-2)^{i+1}}{i+1}x^{i+1} = \sum_{i=0}^{\infty} \frac{(-1)2^{i+1}}{i+1}x^{i+1}$$

$$\ln(1+2x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1}(2x)^{i+1} = \sum_{i=0}^{\infty} \frac{(-1)^i 2^{i+1}}{i+1}x^{i+1}$$

and so

$$\begin{aligned}
\ln\left(\frac{1-2x}{1+2x}\right) &= \sum_{i=0}^{\infty} \frac{(-1)2^{i+1}}{i+1}x^{i+1} - \sum_{i=0}^{\infty} \frac{(-1)^i 2^{i+1}}{i+1}x^{i+1} \\
&= \sum_{i=0}^{\infty} \frac{(-1)2^{i+1} - (-1)^i 2^{i+1}}{i+1}x^{i+1} \\
&= \sum_{i=0}^{\infty} \frac{2^{i+1}[-1-(-1)^i]}{i+1}x^{i+1} \\
&= \sum_{i=0}^{\infty} \frac{2^{i+1}[-1+(-1)^{i+1}]}{i+1}x^{i+1}.
\end{aligned}$$