MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 0 | **MATH 2000** | Fall 2018

SOLUTIONS

[3] 1. We expand and identify the highest power of x in the denominator:

$$\begin{aligned}\lim_{x\to\infty}\frac{6x^3(x-5)}{(3x^2+4)^2} &= \lim_{x\to\infty}\frac{6x^4-30x^3}{9x^4+24x^2+16}\cdot\frac{\frac{1}{x^4}}{\frac{1}{x^4}}\\ &= \lim_{x\to\infty}\frac{6-\frac{30}{x}}{9+\frac{24}{x^2}+\frac{16}{x^4}}\\ &= \frac{6-0}{9+0+0}\\ &= \frac{2}{3}.\end{aligned}$$

[5] 2. Observe that this is a 1^∞ indeterminate form, so first let

$$y=\left(1+\frac{7}{x}\right)^{2x}\quad\text{so}\quad \ln(y)=2x\ln\left(1+\frac{7}{x}\right)=\frac{2\ln\left(1+\frac{7}{x}\right)}{\frac{1}{x}}.$$

Now we have a $\frac{0}{0}$ indeterminate form, so we can apply l'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to\infty}\ln(y) &\overset{\text{H}}{=} \lim_{x\to\infty}\frac{2\cdot\frac{x}{x+7}\cdot-\frac{7}{x^2}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{14x}{x+7}\\ &= 14.\end{aligned}$$

Hence $\lim\limits_{x\to\infty} y = e^{14}$.

[3] 3. (a) We use the Quotient Rule, followed by the Product Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x+xe^x)(x^3+1)-3x^2(xe^x)}{(x^3+1)^2}\\ &= \frac{x^4e^x-2x^3e^x+xe^x+e^x}{(x^3+1)^2}.\end{aligned}$$

[3] (b) We use the Chain Rule (three times):

$$\begin{aligned}\frac{dy}{dx} &= 4\csc^3\left(\sqrt{\tan(x)}\right)\cdot\left[-\csc\left(\sqrt{\tan(x)}\right)\cot\left(\sqrt{\tan(x)}\right)\right]\cdot\frac{1}{2\sqrt{\tan(x)}}\cdot\sec^2(x)\\ &= \frac{-2\csc^4\left(\sqrt{\tan(x)}\right)\cot\left(\sqrt{\tan(x)}\right)\sec^2(x)}{\sqrt{\tan(x)}}.\end{aligned}$$

[4] (c) We differentiate implicitly:

$$\begin{aligned}
\frac{d}{dx}[\sinh(x) + 3y] &= \frac{d}{dx}[x^2\cos(y)] \\
\cosh(x) + 3\frac{dy}{dx} &= 2x\cos(y) - x^2\sin(y)\frac{dy}{dx} \\
\frac{dy}{dx}[3 + x^2\sin(y)] &= 2x\cos(y) - \cosh(x) \\
\frac{dy}{dx} &= \frac{2x\cos(y) - \cosh(x)}{3 + x^2\sin(y)}.
\end{aligned}$$

[4] 4. (a) We use integration by parts, with $w = \ln(x)$ so $dw = \frac{1}{x}\,dx$, and $dv = x^3\,dx$ so $v = \frac{1}{4}x^4$. Then

$$\begin{aligned}
\int x^3\ln(x)\,dx &= vw - \int v\,dw \\
&= \frac{1}{4}x^4\ln(x) - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx \\
&= \frac{1}{4}x^4\ln(x) - \frac{1}{4}\int x^3\,dx \\
&= \frac{1}{4}x^4\ln(x) - \frac{1}{4}\cdot\frac{1}{4}x^4 + C \\
&= \frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4 + C.
\end{aligned}$$

[5] (b) We use the method of partial fractions, first observing that

$$x^3 + x^2 + 4x + 4 = x^2(x+1) + 4(x+1) = (x+1)(x^2+4)$$

so

$$\begin{aligned}
\frac{3x^2 - 8x + 4}{x^3 + x^2 + 4x + 4} &= \frac{A}{x+1} + \frac{Bx + C}{x^2+4} \\
3x^2 - 8x + 4 &= A(x^2+4) + (Bx+C)(x+1).
\end{aligned}$$

When $x = -1$, we have $15 = 5A$ so $A = 3$. When $x = 0$, we have $4 = 4A + C = 12 + C$ so $C = -8$. When (say) $x = 1$, we have $-1 = 5A + 2B + 2C = 2B - 1$ so $B = 0$. Thus

$$\begin{aligned}
\int \frac{3x^2 - 8x + 4}{x^3 + x^2 + 4x + 4}\,dx &= \int \left[\frac{3}{x+1} - \frac{8}{x^2+4}\right] dx \\
&= 3\ln|x+1| - 8\cdot\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C \\
&= 3\ln|x+1| - 4\arctan\left(\frac{x}{2}\right) + C.
\end{aligned}$$

[3] (c) We use u-substitution with $u = 16 - x^2$ so $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. Then

$$\begin{aligned}\int x\sqrt{16-x^2}\,dx &= -\frac{1}{2}\int u^{\frac{1}{2}}\,du\\ &= -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C\\ &= -\frac{1}{3}(16-x^2)^{\frac{3}{2}} + C.\end{aligned}$$

[5] (d) We use trigonometric substitution with $x = 4\sin(\theta)$ so $dx = 4\cos(\theta)\,d\theta$. Then

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2(\theta)} = \sqrt{16\cos^2(\theta)} = 4\cos(\theta)$$

and so the integral becomes

$$\begin{aligned}\int \sqrt{16-x^2}\,dx &= \int 4\cos(\theta)\cdot 4\cos(\theta)\,d\theta\\ &= 16\int \cos^2(\theta)\,d\theta\\ &= 16\int \frac{1+\cos(2\theta)}{2}\,d\theta\\ &= 8\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C\\ &= 8\theta + 8\sin(\theta)\cos(\theta) + C\\ &= 8\arcsin\left(\frac{x}{4}\right) + 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C\\ &= 8\arcsin\left(\frac{x}{4}\right) + \frac{1}{2}x\sqrt{16-x^2} + C.\end{aligned}$$

[5] 5. First we complete the square:

$$\begin{aligned}12x - 9x^2 &= -9\left[x^2 - \frac{4}{3}x\right]\\ &= -9\left[\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) - \frac{4}{9}\right]\\ &= -9\left[\left(x - \frac{2}{3}\right)^2 - \frac{4}{9}\right]\\ &= 4 - 3^2\left(x - \frac{2}{3}\right)^2\\ &= 4 - (3x-2)^2.\end{aligned}$$

Thus

$$\int_{\frac{2}{3}}^{1} \frac{1}{\sqrt{12x-9x^2}}\,dx = \int_{\frac{2}{3}}^{1} \frac{1}{4-(3x-2)^2}\,dx.$$

Now we let $u = 3x-2$ so $\frac{1}{3}\,du = dx$. When $x = \frac{2}{3}$, $u = 0$. When $x = 1$, $u = 1$. The integral becomes

$$\begin{aligned}
\int_{\frac{2}{3}}^{1} \frac{1}{\sqrt{12x-9x^2}}\,dx &= \frac{1}{3}\int_{0}^{1} \frac{1}{\sqrt{4-u^2}}\,du \\
&= \frac{1}{3}\left[\arcsin\left(\frac{u}{2}\right)\right]_0^1 \\
&= \frac{1}{3}\left[\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right] \\
&= \frac{1}{3}\left[\frac{\pi}{6} - 0\right] \\
&= \frac{\pi}{18}.
\end{aligned}$$