Testing the Convergence of Series: A Strategy Guide

Compared to choosing a technique for integrating a function, or finding an approach to compute the limit of a sequence, determining a way to test whether a series $\sum a_i$ converges or diverges can seem even more challenging. That's because these tests tend to apply to a broad range of series, often making an immediate selection difficult.

The best way to identify the right test is often to consider the form of a_i. The following set of questions should keep you on the right track, but don't be reluctant to try more than one approach. Some series can be analysed via multiple methods, and sometimes an apparently appropriate choice just won't work out in certain situations.

- **Is the series familiar?** Is it a known type, like a p-series or a geometric series? If not, can it be written in terms of these series? In either case, we can use the basic results about these common series to test for convergence.

- **Can we easily apply the Divergence Test?** If we can straightforwardly show that $\lim_{i\to\infty} a_i \neq 0$ then we can immediately conclude that the series diverges. Just remember that if $\lim_{i\to\infty} a_i = 0$ then we can draw no conclusion — the Divergence Test *cannot* be used to demonstrate convergence!

- **Does the series consist of products and/or quotients?** If so, the Ratio or Root Tests may be appropriate, especially if the series consists of both positive and negative terms. (These series can sometimes be applied if the series includes sums or differences, but their presence often makes the necessary limits difficult to compute.)

  - The Ratio Test works best if a_i involves factorials and/or simple exponential expressions.
  - The Root Test is more appropriate if a_i involves more complicated exponential expressions, such as those of the form $[f(i)]^i$.

- **Is the series alternating?** If so, the Alternating Series Test is often useful. The Alternating Series Test is so simple that it's sometimes preferable to the Ratio or Root Tests, even in cases where one or both of these could be applied.

- **Does the series consist of positive and negative terms?** If so, consider the corresponding absolute series and, if it converges (by one of the tests for positive series listed below), apply the Absolute Series Test. Alternating series can often be tested this way, although it's usually simpler to apply the Ratio Test, Root Test or Alternating Series Test.

- **Is the series positive?** If so, try the Direct Comparison Test or the Limit Comparison Test, as long as the series resembles a geometric series or a p-series as i becomes large; this will then be the appropriate choice for the test series. If no such test series seems applicable, try the Integral Test, as long as all the relevant conditions are met.