Limits of Sequences: A Strategy Guide

With six methods for investigating the convergence or divergence of a sequence $\{a_i\}$, it's easy to lose track of when each one is relevant and how they are applied. This short summary is designed to be an at-a-glance guide to each approach.

1. Basic limit properties and familiar sequences

   - **Basic Idea:** If a_i consists of common sequences (such as p-sequences or geometric sequences) or other expressions whose limits are obvious, we can evaluate the limit directly
   - **Typical Example**: $\lim\limits_{i\to\infty} \dfrac{2^i+5}{6^i-4^i}$
   - **Demonstrates Convergence or Divergence or Both?** Both
   - **Compute the Limit if Convergent?** Yes
   - **Potential Pitfall:** Remember that the basic limit properties only apply to convergent sequences. For example, we cannot write

     $$\lim_{i\to\infty} \frac{(-1)^i}{i} = \left[\lim_{i\to\infty} (-1)^i\right] \cdot \left[\lim_{i\to\infty} \frac{1}{i}\right]$$

     because $\lim\limits_{i\to\infty} (-1)^i$ does not exist.

2. Rewriting sequences

   - **Basic Idea:** Sometimes a_i is not given in terms of familiar expressions but can be rewritten as such
   - **Typical Example**: $\lim\limits_{i\to\infty} \dfrac{\sin(2i)}{i\sin(i)\cos(i)}$
   - **Demonstrates Convergence or Divergence or Both?** Both
   - **Compute the Limit if Convergent?** Yes
   - **Potential Pitfall:** See Method #1

3. The Evaluation Theorem and l'Hôpital's Rule

   - **Basic Idea:** We cannot apply l'Hôpital's Rule directly to $\lim\limits_{i\to\infty} a_i$ but we can apply it to $\lim\limits_{x\to\infty} f(x)$ where $f(i) = a_i$
   - **Typical Example**: $\lim\limits_{i\to\infty} \dfrac{\ln(i)}{i}$
   - **Demonstrates Convergence or Divergence or Both?** Convergence only (exceptions exist, but we did not cover them)
   - **Compute the Limit if Convergent?** Yes
   - **Potential Pitfall:** If l'Hôpital's Rule shows that $\lim\limits_{x\to\infty} f(x)$ does not exist, we cannot conclude that $\{a_i\}$ is divergent (although this *may* be true)

4. The Squeeze Theorem

- **Basic Idea:** We find two sequences $\{b_i\}$ and $\{c_i\}$ with the same limit and for which $c_i \leq a_i \leq b_i$
- **Typical Example**: $\lim_{i\to\infty} \frac{\sin(i)}{i}$
- **Demonstrates Convergence or Divergence or Both?** Convergence only (exceptions exist, but we did not cover them)
- **Compute the Limit if Convergent?** Yes
- **Potential Pitfall:** Both parts of the Squeeze Theorem must hold, which can make identifying appropriate sequences $\{b_i\}$ and $\{c_i\}$ very challenging

5. The Absolute Sequence Theorem

- **Basic Idea:** If we can show that $\lim_{i\to\infty} |a_i| = 0$ then $\lim_{i\to\infty} a_i = 0$ as well
- **Typical Example**: $\lim_{i\to\infty} (-1)^i \frac{1}{i}$
- **Demonstrates Convergence or Divergence or Both?** Convergence only (but see **Potential Pitfall** below)
- **Compute the Limit if Convergent?** Yes
- **Potential Pitfall:** The Absolute Sequence Theorem does not apply if $\lim_{i\to\infty} |a_i| \neq 0$, but we can sometimes use this result to argue why $\{a_i\}$ must diverge

6. The Bounded Monotonic Sequence Theorem (BMST)

- **Basic Idea:** We show that $\{a_i\}$ is monotonic (or has a monotonic tail) and is bounded
- **Typical Example**: $\lim_{i\to\infty} \frac{2^i}{i!}$
- **Demonstrates Convergence or Divergence or Both?** Convergence only
- **Compute the Limit if Convergent?** No
- **Potential Pitfall:** The BMST can tell us that a sequence is convergent, but cannot be used to determine the value of the limit