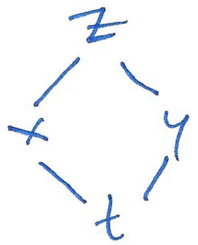
Section 2.4: The Chain Rule

Given a function $y = f(u)$ where $u = g(x)$ the Chain Rule for single-variable functions states that

$$y' = f'(u)u' = f'(g(x))g'(x) \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

So now consider a function $z = f(x,y)$ where $x = g(t)$ and $y = h(t)$.



eg Suppose a cylindrical ice cube is melting. Find an expression for the rate of change of its volume.

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi \cdot 2r \frac{dr}{dt} \cdot h + \pi r^2 \cdot \frac{dh}{dt}$$

$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

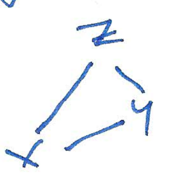
But observe that

$$\frac{\partial V}{\partial r} = 2\pi r h \quad \text{and} \quad \frac{\partial V}{\partial h} = \pi r^2$$

so $$\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$

In general, $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Now consider the special case of a function $z = f(x,y)$ where $y = g(x)$.



$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

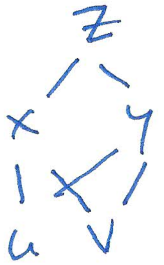
eg $z = \sin(y)\tan(x)$ where $y = \ln(x)$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$= \sin(y)\sec^2(x) + \cos(y)\tan(x)\cdot\frac{1}{x}$$

$$= \sin(y)\sec^2(x) + \frac{\cos(y)\tan(x)}{x}$$

Now consider a function $z = f(x,y)$ where $x = g(u,v)$ and $y = h(u,v)$.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

eg $z = p^2 - q^2$ where $p = r\cos(\theta)$ and $q = r\sin(\theta)$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial p}\cdot\frac{\partial p}{\partial r} + \frac{\partial z}{\partial q}\cdot\frac{\partial q}{\partial r}$$

$$= 2p\cdot\cos(\theta) + (-2q)\cdot\sin(\theta)$$

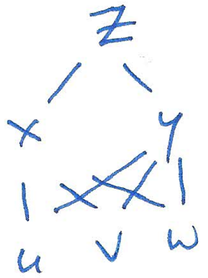
$$= 2p\cos(\theta) - 2q\sin(\theta)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial p}\cdot\frac{\partial p}{\partial \theta} + \frac{\partial z}{\partial q}\cdot\frac{\partial q}{\partial \theta}$$

$$= 2p\cdot[-r\sin(\theta)] + (-2q)\cdot r\cos(\theta)$$

$$= -2pr\sin(\theta) - 2qr\cos(\theta)$$

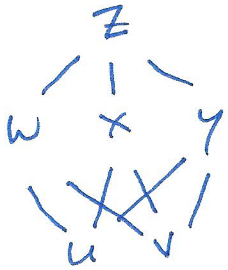
Next suppose $z = f(x,y)$ where $x = g(u,v,w)$, and $y = h(u,v,w)$.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

and similarly for $\frac{\partial z}{\partial v}$ and $\frac{\partial z}{\partial w}$

Conversely, we could have $z = f(w,x,y)$ where $w = g(u,v)$, $x = h(u,v)$ and $y = i(u,v)$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial u} + \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

and likewise for $\frac{\partial z}{\partial v}$.

Theorem: General Chain Rule

Suppose that z is a differentiable function of n variables $u_1, u_2, u_3, \dots, u_n$ and each of these is a differentiable function of m variables $x_1, x_2, x_3, \dots, x_m$. Then

$$\frac{\partial z}{\partial x_i} = \frac{\partial z}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_i} + \frac{\partial z}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_i} + \dots + \frac{\partial z}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_i}$$

for each $i = 1, 2, 3, 4, \dots, m$.

Now suppose we have an equation which expresses y as an implicit function of x.

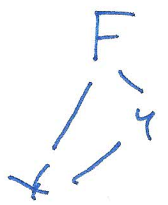
$$\text{eg } x^3 + y^3 = 6xy$$

This can always be written in the form $F(x,y) = 0$ by bringing all terms to one side of the equation.

eg We can write $x^3+y^3=6xy$ as

$$x^3+y^3-6xy=0$$

so $F(x,y) = x^3+y^3-6xy.$



Now the General Chain Rule states that

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

But since $F(x,y)=0$ we have $\frac{dF}{dx}=0$ so

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y} = -\frac{F_x}{F_y}$$

eg For $x^3+y^3=6xy$,

$$\frac{\partial F}{\partial x} = 3x^2-6y \quad \text{and} \quad \frac{\partial F}{\partial y} = 3y^2-6x$$

so $$\frac{dy}{dx} = -\frac{3x^2-6y}{3y^2-6x} \boxed{= \frac{2y-x^2}{y^2-2x}}$$