Section 2.6: Double Integrals over Rectangles

Recall that for a single-variable function $f(x)$,

$$\int f(x)\,dx = F(x) + C$$

where $F'(x) = f(x)$ and C is an arbitrary constant.

Now, given a function $f(x,y)$, we define

$$\int f(x,y)\,dx = F(x,y) + C(y)$$

where $F_x(x,y) = f(x,y)$ and $C(y)$ is an arbitrary function of y

$$\int f(x,y)\,dy = G(x,y) + D(x)$$

where $G_y(x,y) = f(x,y)$ and $D(x)$ is an arbitrary function of x.

These are called <u>partial integrals</u>.

eg $f(x,y) = x^2\cos(y)$

$$\int f(x,y)\,dx = \int x^2\cos(y)\,dx = \frac{x^3}{3}\cos(y) + C(y)$$

$$\int f(x,y)\,dy = \int x^2\cos(y)\,dy = x^2\sin(y) + C(x)$$

Likewise, we can apply the Fundamental Theorem of Calculus to evaluate definite integrals.

eg $\displaystyle\int_1^2 (x^2y - 4x)\,dx = \left[\frac{x^3}{3}y - 2x^2\right]_{x=1}^{x=2}$

$$= \left[\frac{8}{3}y - 8\right] - \left[\frac{1}{3}y - 2\right]$$

$$\boxed{= \frac{7}{3}y - 6}$$

eg $\int_1^2 (x^2y - 4x)\,dy = \left[x^2\frac{y^2}{2} - 4xy\right]_{y=1}^{y=2}$

$= \left[2x^2 - 8x\right] - \left[\frac{1}{2}x^2 - 4x\right]$

$\boxed{= \frac{3}{2}x^2 - 4x}$

Observe that $\int_a^b f(x,y)\,dx = g(y)$ while $\int_c^d f(x,y)\,dy = h(x)$.

eg Evaluate $\int_{-3}^{3} g(y)\,dy$ where $g(y) = \int_1^e \frac{y+5}{x}\,dx$.

$g(y) = \int_1^e \frac{y+5}{x}\,dx = \left[(y+5)\ln|x|\right]_{x=1}^{x=e}$

$= \left[(y+5)\ln(e)\right] - \left[(y+5)\ln(1)\right]$

$= (y+5)$

Thus $\int_{-3}^{3} g(y)\,dy = \int_{-3}^{3} (y+5)\,dy$

$= \left[\frac{1}{2}y^2 + 5y\right]_{-3}^{3} \boxed{= 30}$

Now we can write

$$\int_{-3}^{3}\left(\int_1^e \frac{y+5}{x}\,dx\right)dy = 30$$

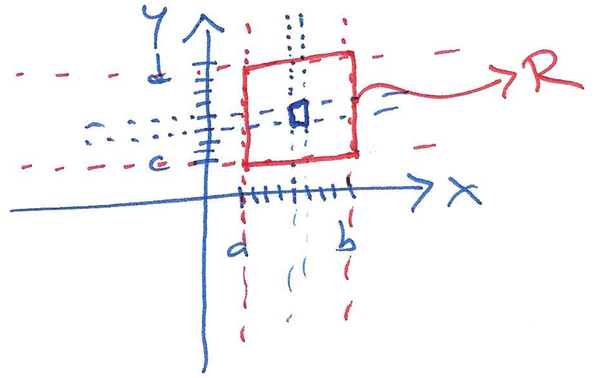
which we can simply write as

$$\int_{-3}^{3}\int_1^e \frac{y+5}{x}\,dx\,dy = 30$$

This is called an <u>iterated</u> integral.

Recall that the graph of $z = f(x,y)$ is a surface. Suppose that $f(x,y) \geq 0$ and we want to find the volume of the region that lies below this surface and above the rectangle R defined by $a \leq x \leq b$ and $c \leq y \leq d$, denoted by

$$R : [a,b] \times [c,d].$$



We will divide $[a,b]$ into a number of sub-intervals, and likewise $[c,d]$. This divides R into a number of <u>sub-rectangles</u>. Then we draw <u>columns</u> from each sub-rectangle to meet the surface $z = f(x,y)$ and compute the volume of each column:

$$\Delta V = f(x,y)\,\Delta A$$

where $\Delta A = \Delta x \Delta y$ is the area of the sub-rectangle.

Then the sum of the volumes of the columns will approximate the true volume V of the region.

$$V \approx \sum_y \sum_x \Delta V = \sum_y \sum_x f(x,y)\,\Delta x$$

If we then allow the number of sub-retangles to become infinite,

$$V = \lim_{n \to \infty} \sum_y \sum_x f(x,y)\,\Delta x$$

Compare from Math 1001: $A = \lim_{n \to \infty} \sum f(x)\,\Delta x = \int_a^b f(x)\,dx$

We write
$$V = \iint_R f(x,y)\,dA$$
called a <u>double integral</u> over the rectangle R.

But observe that
$$\Delta A = \Delta x \Delta y = \Delta y \Delta x$$
$$dA = dx\,dy = dy\,dx$$

Thus
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

and
$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$