Section 1.8

Given a power series in the form $\sum k_i (x-p)^i$ where the formula for k_i is the same for all i then we can define

$$\rho = \lim_{i\to\infty} \left|\frac{k_{i+1}}{k_i}\right| \quad \text{or} \quad \rho = \lim_{i\to\infty} |k_i|^{1/i}$$

Then the radius of convergence $R = 1/\rho$. If $\rho = 0$ we take $R=\infty$ and if $\rho=\infty$ then we take $R=0$.

eg $\displaystyle\sum_{i=0}^{\infty} \frac{x^i}{i+4}$ $\qquad k_i = \frac{1}{i+4}$

centre: $x=0$

$$\rho = \lim_{i\to\infty} \left|\frac{k_{i+1}}{k_i}\right| = \lim_{i\to\infty} \frac{1}{i+5}\cdot(i+4)$$

$$= \lim_{i\to\infty} \frac{i+4}{i+5} = 1 \quad \text{so } R = 1/\rho = 1/1 = 1$$

Thus we know this power series converges for $|x|<1$ or $-1<x<1$.

For $x=1$, the power series becomes $\displaystyle\sum_{i=0}^{\infty} \frac{1}{i+4}$.

We compare with $\sum \frac{1}{i}$ (divergent):

$$\lim_{i\to\infty} \frac{a_i}{t_i} = \lim_{i\to\infty} \frac{1}{i+4}\cdot i = \lim_{i\to\infty} \frac{i}{i+4} = 1$$

so by the LCT, the power series at $x=1$ <u>diverges.</u>

For $x=-1$, the power series becomes $\displaystyle\sum_{i=0}^{\infty} \frac{(-1)^i}{i+4}$.

Here, $\displaystyle\lim_{i\to\infty} p_i = \lim_{i\to\infty} \frac{1}{i+4} = 0$.

Let $f(x) = \frac{1}{x+4}$ so $f'(x) = -\frac{1}{(x+4)^2} < 0$ so $\{p_i\}$ is decreasing. Then by the Alt. Series Test, the power series at $x=-1$ <u>converges.</u>

The interval of convergence is $-1 \le x < 1$.

eg $\sum_{i=0}^{\infty} \frac{i}{3^i}(x-6)^i$ $\quad k_i = \frac{i}{3^i}$

centre: $x=6$

$$\rho = \lim_{i\to\infty} |k_i|^{1/i} = \lim_{i\to\infty} \frac{i^{1/i}}{3} = \frac{1}{3} \quad \text{so } R = 1/\rho = 3$$

The power series converges for $|x-6| < 3$

$$-3 < x-6 < 3$$

$$3 < x < 9$$

For $x=9$, the power series becomes $\sum_{i=0}^{\infty} \frac{i}{3^i}\cdot 3^i = \sum_{i=0}^{\infty} i$

which diverges by the Divergence Test.

For $x=3$, the power series becomes $\sum_{i=0}^{\infty} \frac{i}{3^i}\cdot(-3)^i$

This also diverges by the Divergence Test.

$$= \sum_{i=0}^{\infty} \frac{i}{3^i}\cdot(-1)^i\cdot 3^i$$

$$= \sum_{i=0}^{\infty} (-1)^i i$$

The interval of convergence is $3 < x < 9$.

eg $\sum_{i=1}^{\infty} \frac{(2x-1)^i}{i^2}$

We need to first rewrite the power series:

$$\sum_{i=1}^{\infty} \frac{[2(x-1/2)]^i}{i^2} = \sum_{i=1}^{\infty} \frac{2^i}{i^2}(x-1/2)^i$$

$k_i = \frac{2^i}{i^2}$ $\quad$ centre: $x = 1/2$ (not $x=1$)

$$\rho = \lim_{i\to\infty} |k_i|^{1/i} = \lim_{i\to\infty} \frac{2}{(i^{1/i})^2} = \frac{2}{1^2} = 2 \quad \text{so } R = 1/2$$

The power series converges for $|x-1/2| < 1/2$

$$-\tfrac{1}{2} < x-\tfrac{1}{2} < \tfrac{1}{2}$$

$$0 < x < 1$$

At $x=1$, the power series becomes $\sum_{i=1}^{\infty} \frac{1}{i^2}$ (convergent p-series)
At $x=0$, the power series becomes $\sum_{i=1}^{\infty} \frac{(-1)^i}{i^2}$ which converges by the Abs. Series Test because its absolute series is $\sum \frac{1}{i^2}$.
The interval of convergence is $0 \le x \le 1$.

eg $\sum_{i=0}^{\infty} 4^i x^{2i} = 1 + 4x^2 + 16x^4 + 64x^6 + \ldots$

We have a different formula for k_i when i is even ($k_i = 4^i$) and when i is odd ($k_i = 0$). Thus we <u>cannot</u> use the formulas for ρ.

We must apply the full Root (or Ratio) Test:

$$L = \lim_{i\to\infty} |a_i|^{1/i} = \lim_{i\to\infty} 4x^2 = 4x^2$$

The power series converges when

$$4x^2 < 1$$
$$x^2 < 1/4$$
$$-\tfrac{1}{2} < x < \tfrac{1}{2}$$

This power series has $R = \frac{1}{2}$.

At $x = 1/2$, the power series becomes $\sum_{i=0}^{\infty} 4^i \left(\frac{1}{2}\right)^{2i} = \sum_{i=0}^{\infty} 1$ which diverges by the Divergence Test.

At $x = -1/2$, the power series becomes $\sum_{i=0}^{\infty} 4^i \left(-\frac{1}{2}\right)^{2i} = \sum_{i=0}^{\infty} 1$ which, again, diverges by the Divergence Test.

The interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

Section 1.9: Representing Functions as Power Series

Consider a power series centred at $x=0$ with $k_i = 1$ for all i.

This is given by

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad \text{for } -1 < x < 1$$

This means that the function $\frac{1}{1-x}$ has the <u>power series representation</u> $\sum_{i=0}^{\infty} x^i$ for $-1 < x < 1$.

We can use this result to find power series representations of other functions.

eg $f(x) = \frac{1}{1+x}$

We can convert $\frac{1}{1-x}$ into $\frac{1}{1+x}$ by replacing x with $-x$. Thus we can obtain the power series representation of $\frac{1}{1+x}$ by manipulating $\sum_{i=0}^{\infty} x^i$ in the same way.

Hence

$$\frac{1}{1+x} = \sum_{i=0}^{\infty} (-x)^i$$

$$= \sum_{i=0}^{\infty} (-1)^i x^i$$

This will converge when $-1 < -x < 1$

$$1 > x > -1$$