Section 1.6

If $c = \sqrt[n]{b}$ then we know that $c^n = b$. But then

$$(\sqrt[n]{b})^n = b.$$

Now suppose that $\sqrt[n]{b} = b^p$. Then we have

$$(b^p)^n = b$$

$$b^{p \cdot n} = b^1$$

so $p \cdot n = 1$ so $p = 1/n$. Therefore

$$\sqrt[n]{b} = b^{1/n}.$$

eg Since $\sqrt{64} = 8$ so we can also write $64^{1/2} = 8$.

Since $\sqrt[3]{-64} = -4$ so $(-64)^{1/3} = -4$.

Furthermore, $(\sqrt[n]{b})^m = (b^{1/n})^m = b^{\frac{1}{n} \cdot m} = b^{m/n}$.

eg $9^{3/2} = (9^{1/2})^3 = (\sqrt{9})^3 = 3^3 \boxed{= 27}$

eg $(-8)^{5/3} = \left[(-8)^{1/3}\right]^5 = (\sqrt[3]{-8})^5 = (-2)^5 \boxed{= -32}$

eg $16^{-3/4} = \dfrac{1}{16^{3/4}} = \dfrac{1}{(16^{1/4})^3} = \dfrac{1}{(\sqrt[4]{16})^3}$

$= \dfrac{1}{2^3} \boxed{= \dfrac{1}{8}}$

Properties of radicals

① $\sqrt[n]{b}\cdot\sqrt[n]{c} = \sqrt[n]{bc}$

This must be the case because $\sqrt[n]{b} = b^{1/n}$ and $\sqrt[n]{c} = c^{1/n}$ so

$$\begin{aligned}\sqrt[n]{b}\cdot\sqrt[n]{c} &= b^{1/n}\cdot c^{1/n}\\ &= (bc)^{1/n}\\ &= \sqrt[n]{bc}.\end{aligned}$$

eg $\sqrt{8}\cdot\sqrt{2} = \sqrt{8\cdot 2} = \sqrt{16}\ \boxed{=4}$

② $\dfrac{\sqrt[n]{b}}{\sqrt[n]{c}} = \sqrt[n]{\dfrac{b}{c}}$

eg $\dfrac{\sqrt[3]{x}}{\sqrt[3]{x^5}} = \sqrt[3]{\dfrac{x}{x^5}} = \sqrt[3]{\dfrac{1}{x^4}} = \dfrac{\sqrt[3]{1}}{\sqrt[3]{x^4}}\ \boxed{= \dfrac{1}{\sqrt[3]{x^4}}}$

③ $\sqrt[m]{\sqrt[n]{b}} = \sqrt[m\cdot n]{b}$

eg $\sqrt{\sqrt[3]{x}} = \sqrt[2]{\sqrt[3]{x}} = \sqrt[2\cdot 3]{x}\ \boxed{= \sqrt[6]{x}}$

It is often easier to manipulate radicals by writing them as exponents.

eg $\sqrt{x}\cdot\sqrt[3]{x} = x^{1/2}\cdot x^{1/3} = x^{1/2+1/3}$

$= x^{3/6+2/6}\ \boxed{= x^{5/6}} = \left(\sqrt[6]{x}\right)^5$

eg $\dfrac{6\sqrt[3]{x^4}}{2x^2} = \dfrac{6x^{4/3}}{2x^2} = 3x^{4/3-2} = 3x^{4/3-6/3}$

$$\boxed{= 3x^{-2/3} = \frac{3}{x^{2/3}}}$$

We can extend our idea of factoring to include expressions involving radicals.

eg Factor $4x^{7/2} + 8x^{5/2} + 4x^{3/2}$.

We extract the largest common factor for which a polynomial will be left:

$$4x^{3/2}(x^2 + 2x + 1)$$

$$= 4x^{3/2}(x+1)(x+1)$$

$$\boxed{= 4x^{3/2}(x+1)^2}$$

We want to simplify radicals so that the argument of the radical (the expression inside the root) is reduced to the smallest possible terms.

eg $\sqrt{32} = \sqrt{4\cdot 4\cdot 2} = \sqrt{16\cdot 2} = \sqrt{16}\cdot\sqrt{2}\ \boxed{= 4\sqrt{2}}$

eg $\sqrt{x^2} \neq x$

In fact, $$\sqrt{x^2} = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$$

$$= |x|.$$

eg Assuming $x \geq 0$, simplify $\sqrt{50x^5} + \sqrt{18x^3}$.

Here, $\sqrt{50x^5} = \sqrt{25x^4 \cdot 2x} = \sqrt{25x^4} \cdot \sqrt{2x}$

$= 5x^2\sqrt{2x}$

$\sqrt{18x^3} = \sqrt{9x^2 \cdot 2x} = \sqrt{9x^2} \cdot \sqrt{2x}$

$= 3x\sqrt{2x}$

Thus $\sqrt{50x^5} + \sqrt{18x^3} = 5x^2\sqrt{2x} + 3x\sqrt{2x}$

$= x\sqrt{2x}(5x+3)$

$= x\sqrt{2} \cdot \sqrt{x}(5x+3)$

$\boxed{= \sqrt{2}\, x^{3/2}(5x+3)}$.

To remove a constant radical from the denominator of an expression, we rationalise the denominator.

First assume that the denominator consists of a single term of the form $k\sqrt{b}$. Then we multiply both the numerator and the denominator by $\sqrt{b}$.

eg Rationalise $\frac{9}{2\sqrt{3}}$.

We write $\frac{9}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{2 \cdot 3}$

$$\boxed{= \frac{3\sqrt{3}}{2}}$$