Section 1.3

We can sometimes factor polynomials using a special formula:

(1) $B^2 - C^2 = (B-C)(B+C)$ ("difference of squares")

eg $x^2 - 9 = x^2 - 3^2$

$\boxed{= (x-3)(x+3)}$

There is no "sum of squares" formula for $B^2 + C^2$.

eg $81x^4 - 16 = (9x^2)^2 - 4^2$

$= (9x^2-4)(9x^2+4)$

We cannot factor $9x^2+4$ because it is a sum of squares. However,

$9x^2 - 4 = (3x)^2 - 2^2$

$= (3x-2)(3x+2)$

Thus $81x^4 - 16 \boxed{= (3x-2)(3x+2)(9x^2+4)}$.

(2) $B^3 - C^3 = (B-C)(B^2 + BC + C^2)$

eg $x^3 - 8 = x^3 - 2^3$

$= (x-2)(x^2 + x \cdot 2 + 2^2)$

$\boxed{= (x-2)(x^2+2x+4)}$

(3) $B^3 + C^3 = (B+C)(B^2 - BC + C^2)$

eg $$64x^3 + 1 = (4x)^3 + 1^3$$
$$= (4x+1)((4x)^2 - 4x \cdot 1 + 1^2)$$
$$\boxed{= (4x+1)(16x^2 - 4x + 1)}$$

To understand how we might factor a trinomial, consider a polynomial of the form

$$x^2 + Bx + C.$$

This must factor into first-degree polynomials $x+p$ and $x+q$.

Then $$x^2 + Bx + C = (x+p)(x+q)$$
$$= x(x+q) + p(x+q)$$
$$= x^2 + qx + px + pq$$
$$= x^2 + (p+q)x + pq.$$

Hence p and q must be chosen so that their product is C and their sum is B.

eg Factor $x^2 + 5x + 6$.

The integers whose product is 6 are:

$1 \cdot 6 = 6$	$1 + 6 = 7$
$-1 \cdot (-6) = 6$	$-1 + (-6) = -7$
$2 \cdot 3 = 6$	$2 + 3 = 5$
$-2 \cdot (-3) = 6$	$-2 + (-3) = -5$

Thus $x^2+5x+6=(x+2)(x+3)$.

eg Factor x^2-x-12.

The integers whose product is -12 are:

$1\cdot(-12)=-12 \qquad 1+(-12)=-11$

$-1\cdot 12=-12 \qquad -1+12=11$

$2\cdot(-6)=-12 \qquad 2+(-6)=-4$

$-2\cdot 6=-12 \qquad -2+6=4$

$3\cdot(-4)=-12 \qquad 3+(-4)=-1$

$-3\cdot 4=-12 \qquad -3+4=1$

Thus $x^2-x-12=(x+3)(x-4)$.

Now suppose that the trinomial has the form

$$Ax^2+Bx+C$$

for $A\neq 1$. Then we have

$$\begin{aligned}Ax^2+Bx+C &= (kx+p)(lx+q)\\ &= kx(lx+q)+p(lx+q)\\ &= (kl)x^2+(kq)x+(lp)x+pq\\ &= (kl)x^2+(kq+lp)x+pq.\end{aligned}$$

This means that we still have $p\cdot q=C$. Furthermore we also have $k\cdot l=A$. Finally, $kq+lp=B$.

eg Factor $4x^2 - 8x + 3$.

The integers whose product is 3 are:

$1 \cdot 3 = 3 \qquad -1 \cdot (-3) = 3$

Thus either $4x^2 - 8x + 3 = (kx+1)(lx+3)$

or $4x^2 - 8x + 3 = (kx-1)(lx-3)$

The integers whose product is 4 are:

$1 \cdot 4 = 4 \qquad \left[\begin{array}{l} -1 \cdot (-4) = 4 \\ -2 \cdot (-2) = 4 \end{array}\right]$

$2 \cdot 2 = 4$

Thus we may have

$$\begin{aligned} 4x^2 - 8x + 3 &= (x+1)(4x+3) = 4x^2 + 7x + 3 \\ &= (x-1)(4x-3) = 4x^2 - 7x + 3 \\ &= (4x+1)(x+3) = 4x^2 + 13x + 3 \\ &= (4x-1)(x-3) = 4x^2 - 13x + 3 \\ &= (2x+1)(2x+3) = 4x^2 + 8x + 3 \\ &= (2x-1)(2x-3) = 4x^2 - 8x + 3 \end{aligned}$$

Finally, $4x^2 - 8x + 3 = (2x-1)(2x-3)$.

eg Factor $-4x^2 + 8x - 3$.

We can rewrite this as $-4x^2 + 8x - 3 = -(4x^2 - 8x + 3)$.

Hence $-4x^2 + 8x - 3 = -(2x-1)(2x-3)$.