Section 1.3: Polynomials

Def'n: A polynomial is any expression which can be written in the form

$$k_n x^n + k_{n-1} x^{n-1} + k_{n-2} x^{n-2} + \dots + k_2 x^2 + k_1 x + k_0$$

where the natural number n is the degree of the polynomial, and the real numbers $k_0, k_1, \dots, k_n$ are its coefficients.

The constant k_0 is the constant coefficient.

The constant k_n is the leading coefficient.

eg $2x+1$ is a polynomial of degree 1 (a "linear" polynomial)

eg $5-4x^2+x = -4x^2+x+5$

is a polynomial of degree 2

(a "quadratic" polynomial)

eg $x^3+\frac{1}{4}x^2-\sqrt{2} = x^3+\frac{1}{4}x^2+0x-\sqrt{2}$

is a polynomial of degree 3

(a "cubic" polynomial)

A constant can be considered a polynomial of degree 0.

A polynomial with exactly two terms is a binomial.

A polynomial with exactly three terms is a trinomial.

We can add or subtract polynomials by adding or subtracting like terms.

eg $(x^2-2x+4)+(x^2+5x-3)$

$= (x^2+x^2)+(-2x+5x)+(4-3)$

$\boxed{= 2x^2+3x+1}$

eg $(2x+7)-(3x^2-8x+7)$

$= (0x^2-3x^2)+(2x-(-8x))+(7-7)$

$\boxed{= -3x^2+10x}$

To multiply polynomials, we apply the property of distributivity.

eg $4x\left(x^2-\frac{1}{2}x+4\right) = 4x\cdot x^2+4x\cdot\left(-\frac{1}{2}x\right)+4x\cdot 4$

$\boxed{= 4x^3-2x^2+16x}$

eg $(3x+2)(x-7) = 3x(x-7)+2(x-7)$

$= 3x\cdot x+3x\cdot(-7)+2\cdot x+2\cdot(-7)$

$= 3x^2-21x+2x-14$

$\boxed{= 3x^2-19x-14}$

Some common products include:

1. $$(B+C)(B-C) = B(B-C) + C(B-C)$$
$$= B\cdot B + B\cdot(-C) + C\cdot B + C\cdot(-C)$$
$$= B^2 - BC + BC - C^2$$
$$= B^2 - C^2$$

eg $$(4x-3)(4x+3) = (4x)^2 - 3^2$$
$$\boxed{= 16x^2 - 9}$$

2. $$(B+C)^2 = (B+C)(B+C)$$
$$= B(B+C) + C(B+C)$$
$$= B^2 + BC + BC + C^2$$
$$= B^2 + 2BC + C^2$$

Observe that $(B+C)^2 \neq B^2 + C^2$

3. $$(B-C)^2 = B^2 - 2BC + C^2$$

eg $$(x^2-5)^2 = (x^2)^2 - 2x^2\cdot 5 + 5^2$$
$$\boxed{= x^4 - 10x^2 + 25}$$

eg $$(x+2)^3 = (x+2)^2(x+2)$$
$$= (x^2 + 2x\cdot 2 + 2^2)(x+2)$$
$$= (x^2 + 4x + 4)(x+2)$$
$$= x^2(x+2) + 4x(x+2) + 4(x+2)$$
$$= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8$$
$$\boxed{= x^3 + 6x^2 + 12x + 8}$$

When a polynomial has integer coefficients, it can be <u>factored</u> by rewriting it as a product of lower-degree polynomials with integer coefficients.

eg $x^2 - 9 = (x-3)(x+3)$

eg $x^2 - 2 = (x-\sqrt{2})(x+\sqrt{2})$ but this is not its factored form because not all the coefficients are integers. Instead, we would consider x^2-2 to already be <u>factored completely</u>.

The simplest way to factor a polynomial is to identify an expression which appears in or is common to every term.

eg $9x^3 - 12x^2 = 3x^2 \cdot 3x - 3x^2 \cdot 4$

$$\boxed{= 3x^2(3x-4)}$$