MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 5.1 **Math 1090** FALL 2009

SOLUTIONS

1. (a) Using long division of polynomials, we have

$$\begin{array}{r@{}l} & 3x^2 + 2x - 3 \\ x^2 - 2x + 3 \,\big) & 3x^4 - 4x^3 + 2x^2 + 10x - 2 \\ & \underline{3x^4 - 6x^3 + 9x^2} \\ & \qquad 2x^3 - 7x^2 + 10x - 2 \\ & \qquad \underline{2x^3 - 4x^2 + 6x} \\ & \qquad\qquad -3x^2 + 4x - 2 \\ & \qquad\qquad \underline{-3x^2 + 6x - 9} \\ & \qquad\qquad\qquad -2x + 7 \end{array}$$

Hence $Q(x) = 3x^2 + 2x - 3$ and $R(x) = -2x + 7$.

(b) By long division of polynomials, we have

$$\begin{array}{r@{}l} & 3x^3 - 7x^2 + 9x + 1 \\ x + 1 \,\big) & 3x^4 - 4x^3 + 2x^2 + 10x - 2 \\ & \underline{3x^4 + 3x^3} \\ & \quad -7x^3 + 2x^2 + 10x - 2 \\ & \quad \underline{-7x^3 - 7x^2} \\ & \qquad\quad 9x^2 + 10x - 2 \\ & \qquad\quad \underline{9x^2 + 9x} \\ & \qquad\qquad\quad x - 2 \\ & \qquad\qquad\quad \underline{x + 1} \\ & \qquad\qquad\qquad -3 \end{array}$$

Hence $Q(x) = 3x^3 - 7x^2 + 9x + 1$ and $R(x) = -3$.

2. (a) By synthetic division with $r = -3$, we have

$$\begin{array}{r|rrrr} -3 & 2 & 3 & -1 & 5 \\ & & -6 & 9 & -24 \\ \hline & 2 & -3 & 8 & -19 \end{array}$$

so $Q(x) = 2x^2 - 3x + 8$ and $R = -19$.

(b) By synthetic division with $r = 2$, we have

$$\begin{array}{r|rrrrr} 2 & 2 & 0 & -11 & 8 & 3 \\ & & 4 & 8 & -6 & 4 \\ \hline & 2 & 4 & -3 & 2 & 7 \end{array}$$

so $Q(x) = 2x^3 + 4x^2 - 3x + 2$ and $R = 7$.

3. (a) The possible rational roots are ±1, ±2, ±5 and ±10. Note that $P(1) = -24$ but $P(-1) = 0$ so -1 is a root of $P(x)$ and $(x+1)$ is a factor. By synthetic division, we have

$$\begin{array}{r|rrrrr} -1 & 1 & 5 & -3 & -17 & -10 \\ & & -1 & -4 & 7 & 10 \\ \hline & 1 & 4 & -7 & -10 & 0 \end{array}$$

so $P(x) = (x+1)(x^3 + 4x^2 - 7x - 10)$. Now for $Q(x) = x^3 + 4x^2 - 7x - 10$ the possible rational roots are -1, ±2, ±5 and ±10. We have $Q(-1) = 0$ so -1 is again a root of $P(x)$ and $(x+1)$ is again a factor. By synthetic division, we have

$$\begin{array}{r|rrrr} -1 & 1 & 4 & -7 & -10 \\ & & -1 & 3 & 10 \\ \hline & 1 & 3 & -10 & 0 \end{array}$$

and so now

$$P(x) = (x+1)^2(x^2 + 3x - 10) = (x+1)^2(x+5)(x-2).$$

Hence the roots of $P(x)$ are -1, -5 and 2.

(b) The possible rational roots of $P(x)$ are ±1, ±2, ±4, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$ and $\pm\frac{1}{6}$. Note that $P(1) = -5$ and $P(-1) = -9$ but $P(2) = 0$. Thus 2 is a root of $P(x)$ and $(x-2)$ is a factor. By synthetic division, we have

$$\begin{array}{r|rrrr} 2 & 6 & -11 & -4 & 4 \\ & & 12 & 2 & -4 \\ \hline & 6 & 1 & -2 & 0 \end{array}$$

so

$$P(x) = (x-2)(6x^2 + x - 2) = (x-2)(3x+2)(2x-1).$$

Hence the roots of $P(x)$ are 2, $-\frac{2}{3}$ and $\frac{1}{2}$.

(c) The possible rational roots of $P(x)$ are ±1, ±2, ±5, ±10, $\pm\frac{1}{2}$ and $\pm\frac{5}{2}$. Note that $P(1) = 6$, $P(-1) = -30$, and $P(2) = 12$ but $P(-2) = 0$. Thus -2 is a root of $P(x)$ and $(x+2)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrr} -2 & 2 & -5 & -4 & 23 & -10 \\ & & -4 & 18 & -28 & 10 \\ \hline & 2 & -9 & 14 & -5 & 0 \end{array}$$

so $P(x) = (x+2)(2x^3 - 9x^2 + 14x - 5)$. The possible rational roots of $Q(x) = 2x^3 - 9x^2 + 14x - 5$ (noting that the coefficients alternate in sign) are 5, $\frac{1}{2}$ and $\frac{5}{2}$. We have $Q(5) = 90$ but $Q\left(\frac{1}{2}\right) = 0$ so $\frac{1}{2}$ is a root of $P(x)$ and $\left(x - \frac{1}{2}\right)$ is a factor. Again using synthetic division we get

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -9 & 14 & -5 \\ & & 1 & -4 & 5 \\ \hline & 2 & -8 & 10 & 0 \end{array}$$

so

$$P(x) = (x+2)\left(x-\frac{1}{2}\right)(2x^2-8x+10) = 2(x+2)\left(x-\frac{1}{2}\right)(x^2-4x+5).$$

By the quadratic formula, if $x^2 - 4x + 5 = 0$ then

$$x = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Hence the roots of $P(x)$ are -2, $\frac{1}{2}$, $2+i$ and $2-i$.

(d) The possible rational roots of $P(x)$ are ± 1, ± 2, ± 4, ± 8 and $\pm\frac{1}{2}$. Note that $P(1) = -3$, $P(-1) = 9$, but $P(2) = 0$. Thus 2 is a root of $P(x)$ and $(x-2)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrrr} 2 & 2 & -7 & 0 & 18 & -8 & -8 \\ & & 4 & -6 & -12 & 12 & 8 \\ \hline & 2 & -3 & -6 & 6 & 4 & 0 \end{array}$$

so $P(x) = (x-2)(2x^4 - 3x^3 - 6x^2 + 6x + 4)$. The possible rational roots of $Q(x) = 2x^4 - 3x^3 - 6x^2 + 6x + 4$ are ± 2, ± 4 and $\pm\frac{1}{2}$. Note that $P(2) = 0$ so 2 is again a root of $P(x)$ and $(x-2)$ is one more a factor. By synthetic division,

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -6 & 6 & 4 \\ & & 4 & 2 & -8 & -4 \\ \hline & 2 & 1 & -4 & -2 & 0 \end{array}$$

so

$$\begin{aligned} P(x) &= (x-2)^2(2x^3 + x^2 - 4x - 2) \\ &= (x-2)^2[x^2(2x+1) - 2(2x+1)] \\ &= (x-2)^2(2x+1)(x^2-2) \\ &= (x-2)^2(2x+1)(x-\sqrt{2})(x+\sqrt{2}), \end{aligned}$$

Hence the roots of $P(x)$ are 2, $-\frac{1}{2}$, $\sqrt{2}$ and $-\sqrt{2}$.

(e) The possible rational roots of $P(x)$ are 1, 2, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$ and $\frac{2}{9}$ (noting that the coefficients alternate in sign and so no negative roots can occur). Note that $P(1) = 12$ and $P(2) = 141$, but $P\left(\frac{1}{3}\right) = 0$. Thus $\frac{1}{3}$ is a root of $P(x)$ are $\left(x - \frac{1}{3}\right)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrr} \frac{1}{3} & 9 & -6 & 19 & -12 & 2 \\ & & 3 & -1 & 6 & -2 \\ \hline & 9 & -3 & 18 & -6 & 0 \end{array}$$

so

$$\begin{aligned} P(x) &= \left(x - \frac{1}{3}\right)(9x^3 - 3x^2 + 18x - 6) \\ &= \left(x - \frac{1}{3}\right)[3x^2(3x-1) + 6(3x-1)] \\ &= \left(x - \frac{1}{3}\right)(3x-1)(3x^2+6) \\ &= 3\left(x - \frac{1}{3}\right)(3x-1)(x^2+2) \\ &= 3\left(x - \frac{1}{3}\right)(3x-1)(x-\sqrt{2}i)(x+\sqrt{2}i). \end{aligned}$$

In fact, note that

$$3\left(x - \frac{1}{3}\right) = 3x - 1$$

so this further simplifies to

$$P(x) = (3x-1)^2(x-2i)(x+2i).$$

Either way, we see that the roots of $P(x)$ are $\frac{1}{3}$, $\sqrt{2}i$ and $-\sqrt{2}i$.

(f) The possible rational roots of $P(x)$ are ± 1, ± 2, ± 4 and ± 8. Note that $P(1) = 20$ and $P(-1) = -36$ but $P(2) = 0$ so 2 is a root of $P(x)$ and $(x-2)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & -17 & 30 & 8 \\ & & 2 & 0 & -34 & -8 \\ \hline & 1 & 0 & -17 & -4 & 0 \end{array}$$

so $P(x) = (x-2)(x^3 - 17x - 4)$. The possible rational roots of $Q(x) = x^3 - 17x - 4$ are ± 2 and ± 4. Note that $Q(2) = -30$, $Q(-2) = 22$ and $Q(4) = -8$, but $Q(-4) = 0$ so -4 is a root of $P(x)$ and $(x+4)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrr} -4 & 1 & 0 & -17 & -4 \\ & & -4 & 16 & 4 \\ \hline & 1 & -4 & -1 & 0 \end{array}$$

so $P(x) = (x-2)(x+4)(x^2 - 4x - 1)$. If $x^2 - 4x - 1 = 0$ then, by the quadratic formula,

$$x = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}.$$

Hence the roots of $P(x)$ are 2, -4, $2+\sqrt{5}$ and $2-\sqrt{5}$.

4. (a) The possible rational roots of $P(x)$ are ± 1, ± 2, ± 4, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, $\pm\frac{1}{8}$ and $\pm\frac{1}{16}$. Note that $P(1) = 15$ and $P(-1) = 135$, but $P(2) = 0$ so 2 is a root of $P(x)$ and $(x-2)$ is a factor.

By synthetic division,

$$\begin{array}{r|rrrrr} 2 & 16 & -64 & 63 & 4 & -4 \\ & & 32 & -64 & -2 & 4 \\ \hline & 16 & -32 & -1 & 2 & 0 \end{array}$$

so

$$\begin{aligned} P(x) &= (x-2)(16x^3 - 32x^2 - x + 2) \\ &= (x-2)[16x^2(x-2) - (x-2)] \\ &= (x-2)(x-2)(16x^2 - 1) \\ &= (x-2)^2(4x-1)(4x+1). \end{aligned}$$

(b) The possible rational roots of $P(x)$ are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12, $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$. Note that $P(1) = -12$ and $P(-1) = 18$, but $P(2) = 0$ so 2 is a root of $P(x)$ and $(x-2)$ is a factor. Then, by synthetic division,

$$\begin{array}{r|rrrrr} 2 & 2 & 5 & -11 & -20 & 12 \\ & & 4 & 18 & 14 & -12 \\ \hline & 2 & 9 & 7 & -6 & 0 \end{array}$$

so $P(x) = (x-2)(2x^3 + 9x^2 + 7x - 6)$. The possible rational roots of $Q(x) = 2x^3 + 9x^2 + 7x - 6$ are ± 2, ± 3, ± 6, $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$. Note that $Q(2) = 60$ but $Q(-2) = 0$ so -2 is a root of $P(x)$ and $(x+2)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrr} -2 & 2 & 9 & 7 & -6 \\ & & -4 & -10 & 6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

so

$$P(x) = (x-2)(x+2)(2x^2 + 5x - 3) = (x-2)(x+2)(2x-1)(x+3).$$

(c) The possible rational roots of $P(x)$ are ± 1, ± 2, ± 4, ± 8, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$ and $\pm\frac{8}{3}$. Note that $P(1) = 30$, $P(-1) = -20$, and $P(2) = 160$, but $P(-2) = 0$ so -2 is a root of $P(x)$ and $(x+2)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrr} -2 & 3 & 5 & 10 & 20 & -8 \\ & & -6 & 2 & -24 & 8 \\ \hline & 3 & -1 & 12 & -4 & 0 \end{array}$$

so

$$\begin{aligned} P(x) &= (x+2)(3x^3 - x^2 + 12x - 4) \\ &= (x+2)[x^2(3x-1) + 4(3x-1)] \\ &= (x+2)(3x-1)(x^2+4) \\ &= (x+2)(3x-1)(x-2i)(x+2i). \end{aligned}$$

(d) The possible rational roots of $P(x)$ are ± 1, ± 2, $\pm\frac{1}{2}$ and $\pm\frac{1}{4}$. Note that $P(1) = 0$ so 1 is a root of $P(x)$ and $(x-1)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrrr} 1 & 4 & 0 & -11 & -1 & 6 & 2 \\ & & 4 & 4 & -7 & -8 & -2 \\ \hline & 4 & 4 & -7 & -8 & -2 & 0 \end{array}$$

so $P(x) = (x-1)(4x^4 + 4x^3 - 7x^2 - 8x - 2)$. The possible rational roots of $Q(x) = 4x^4 + 4x^3 - 7x^2 - 8x - 2$ are ± 1, ± 2, $\pm\frac{1}{2}$ and $\pm\frac{1}{4}$. Note that $Q(1) = -9$, $Q(-1) = -1$, $Q(2) = 50$, $Q(-2) = 18$, and $Q\left(\frac{1}{2}\right) = -7$, but $Q\left(-\frac{1}{2}\right) = 0$ so $-\frac{1}{2}$ is a root of $P(x)$ and $\left(x + \frac{1}{2}\right)$ is a factor. Then by synthetic division,

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 4 & 4 & -7 & -8 & -2 \\ & & -2 & -1 & 4 & 2 \\ \hline & 4 & 2 & -8 & -4 & 0 \end{array}$$

so

$$\begin{aligned} P(x) &= (x-1)\left(x+\frac{1}{2}\right)(4x^3 + 2x^2 - 8x - 4) \\ &= (x-1)\left(x+\frac{1}{2}\right)[2x^2(2x+1) - 4(2x+1)] \\ &= 2(x-1)\left(x+\frac{1}{2}\right)(2x+1)(x^2-2) \\ &= 2(x-1)\left(x+\frac{1}{2}\right)(2x+1)(x-\sqrt{2})(x+\sqrt{2}) \\ &= (x-1)(2x+1)^2(x-\sqrt{2})(x+\sqrt{2}). \end{aligned}$$

5. (a) Solving the given equation is equivalent to finding the roots of the polynomial $P(x) = 2x^4 + 3x^3 - 3x^2 - 7x - 3$. The possible rational roots of $P(x)$ are ± 1, ± 3, $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$. Note that $P(1) = -8$ but $P(-1) = 0$ so -1 is a root of $P(x)$ and $(x+1)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrr} -1 & 2 & 3 & -3 & -7 & -3 \\ & & -2 & -1 & 4 & 3 \\ \hline & 2 & 1 & -4 & -3 & 0 \end{array}$$

so $P(x) = (x+1)(2x^3 + x^2 - 4x - 3)$. The possible rational roots of $Q(x) = 2x^3 + x^2 - 4x - 3$ are -1, ± 3, $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$. Note that $Q(-1) = 0$ so -1 is again a root of $P(x)$ and $(x+1)$ is once more a factor. Then synthetic division yields

$$\begin{array}{r|rrrr} -1 & 2 & 1 & -4 & -3 \\ & & -2 & 1 & 3 \\ \hline & 2 & -1 & -3 & 0 \end{array}$$

so

$$P(x) = (x+1)^2(2x^2 - x - 3) = (x+1)^2(2x-3)(x+1) = (x+1)^3(2x-3).$$

Thus the solutions of the equation are $x = -1$ and $x = \frac{3}{2}$.

(b) Solving the given equation is equivalent to find the roots of the polynomial $P(x) = 8x^5+12x^4+14x^3+13x^2+6x+1$. The possible rational roots of $P(x)$ are -1, $-\frac{1}{2}$, $-\frac{1}{4}$ and $-\frac{1}{8}$ (observing that all of the coefficients of $P(x)$ are positive). Note that $P(-1) = -2$ but $P\left(-\frac{1}{2}\right) = 0$ so $-\frac{1}{2}$ is a root of $P(x)$ and $\left(x+\frac{1}{2}\right)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrrrr} -\frac{1}{2} & 8 & 12 & 14 & 13 & 6 & 1 \\ & & -4 & -4 & -5 & -4 & -1 \\ \hline & 8 & 8 & 10 & 8 & 2 & 0 \end{array}$$

so $P(x) = \left(x+\frac{1}{2}\right)(8x^4+8x^3+10x^2+8x+2)$. The possible rational roots of $Q(x) = 8x^4+8x^3+10x^2+8x+2$ are $-\frac{1}{2}$, $-\frac{1}{4}$ and $-\frac{1}{8}$. Note that $Q\left(-\frac{1}{2}\right) = 0$ so $-\frac{1}{2}$ is at least a double root of $P(x)$ and $\left(x+\frac{1}{2}\right)$ is again a factor. Synthetic division gives

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 8 & 8 & 10 & 8 & 2 \\ & & -4 & -2 & -4 & -2 \\ \hline & 8 & 4 & 8 & 4 & 0 \end{array}$$

so

$$\begin{aligned} P(x) &= \left(x+\frac{1}{2}\right)^2 (8x^3+4x^2+8x+4) \\ &= 4\left(x+\frac{1}{2}\right)^2 (2x^3+x^2+2x+1) \\ &= 4\left(x+\frac{1}{2}\right)^2 [x^2(2x+1)+(2x+1)] \\ &= 4\left(x+\frac{1}{2}\right)^2 (2x+1)(x^2+1) \\ &= 4\left(x+\frac{1}{2}\right)^2 (2x+1)(x-i)(x+i) \\ &= (2x+1)^3(x-i)(x+i). \end{aligned}$$

Hence the solutions of the equation are $x = -\frac{1}{2}$, $x = i$ and $x = -i$.

(c) Solving this equation is equivalent to finding the roots of the polynomial $P(x) = x^3+x-10$. The possible rational roots of $P(x)$ are ±1, ±2, ±5 and ±10. Note that $P(1) = -8$ and $P(-1) = -12$, but $P(2) = 0$ so 2 is a root of $P(x)$ and $(x-2)$ is a factor. By synthetic division,

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 1 & -10 \\ & & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

so $P(x) = (x-2)(x^2+2x+5)$. If we set $x^2+2x+5 = 0$ then the quadratic formula gives

$$x = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm4i}{2} = -1\pm2i.$$

Thus the solutions of the equation are $x = 2$, $x = -1+2i$ and $x = -1-2i$.

6. (a) Such a polynomial is

$$P(x) = (x-2)^3(x+1) = x^4 - 5x^3 + 6x^2 + 4x - 8.$$

(b) Note that if $-1+i$ is a root, then so is $-1-i$. Therefore such a polynomial is

$$P(x) = (x-3)(x-(-1+i))(x-(-1-i)) = (x-3)(x^2+2x+2) = x^3 - x^2 - 4x - 6.$$

(c) First let's just find any polynomial with -4 and $3-2\sqrt{2}$ as roots. Since $3-2\sqrt{2}$ is a root of the polynomial, so too is $3+2\sqrt{2}$. Hence one such polynomial is

$$P_1(x) = (x+4)(x-(3-2\sqrt{2}))(x-(3+2\sqrt{2})) = (x+4)(x^2-6x+1) = x^3-2x^2-23x+4.$$

However, observe that $P_1(0) = 4$. Since we want $P_1(0) = -8 = -2 \cdot 4$, the polynomial we're actually looking for is

$$P(x) = -2P_1(x) = -2x^3 + 4x^2 + 46x - 8.$$