MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.6 **Math 1090** FALL 2009

SOLUTIONS

1. (a) Observe that $315° = 360° - 45°$, so

$$\begin{aligned}\sin(315°) &= \sin(360° - 45°)\\ &= \sin(360°)\cos(45°) - \cos(360°)\sin(45°)\\ &= 0 \cdot \frac{\sqrt{2}}{2} - 1 \cdot \frac{\sqrt{2}}{2}\\ &= -\frac{\sqrt{2}}{2}.\end{aligned}$$

(b) We can write $\dfrac{17\pi}{12} = \dfrac{8\pi}{12} + \dfrac{9\pi}{12} = \dfrac{2\pi}{3} + \dfrac{3\pi}{4}$ so

$$\begin{aligned}\cos\left(\frac{17\pi}{12}\right) &= \cos\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)\\ &= \cos\left(\frac{2\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{3\pi}{4}\right)\\ &= \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}.\end{aligned}$$

(c) Since $\dfrac{3\pi}{8} = \dfrac{1}{2}\left(\dfrac{3\pi}{4}\right)$,

$$\cos\left(\frac{3\pi}{8}\right) = \sqrt{\frac{1 + \cos\left(\frac{3\pi}{4}\right)}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{2}}{4}}.$$

Note that we choose the positive square root because $\frac{3\pi}{8}$ is in the first quadrant, where all trig values are positive.

2. Let's begin by deriving some information we'll use throughout all parts of this question. For α, we can construct a right triangle with opposite sidelength 3, adjacent sidelength 4, and hypotenuse of length

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

Since α is in the third quadrant,

$$\sin(\alpha) = -\frac{3}{5} \quad \text{and} \quad \cos(\alpha) = -\frac{4}{5}.$$

For β, we construct a right triangle with opposite sidelength 5 and hypotenuse of length 13, so its adjacent sidelength is

$$\sqrt{13^2 - 5^2} = \sqrt{144} = 12.$$

Since β is in the fourth quadrant,

$$\cos(\beta) = \frac{12}{13}.$$

(a) $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$

$= -\frac{36}{65} + \frac{20}{65} = -\frac{16}{65}$

(b) $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) = \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$

$= -\frac{36}{65} - \frac{20}{65} = -\frac{56}{65}$

(c) $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right)$

$= -\frac{48}{65} - \frac{15}{65} = -\frac{63}{65}$

(d) $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right)$

$= -\frac{48}{65} + \frac{15}{65} = -\frac{33}{65}$

(e) $\cos\left(\alpha + \frac{\pi}{3}\right) = \cos(\alpha)\cos\left(\frac{\pi}{3}\right) - \sin(\alpha)\sin\left(\frac{\pi}{3}\right) = \left(-\frac{4}{5}\right)\left(\frac{1}{2}\right) - \left(-\frac{3}{5}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{2}{5} + \frac{3\sqrt{3}}{10} = \frac{3\sqrt{3}-4}{10}$

(f) $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) = \frac{24}{25}$

(g) $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = \left(-\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

(h) Since β is in the fourth quadrant, $\frac{1}{2}\beta$ must be in the second quadrant so its sine must be positive, and thus

$$\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{1-\cos(\beta)}{2}} = \sqrt{\frac{1-\frac{12}{13}}{2}} = \sqrt{\frac{1}{26}} = \frac{\sqrt{26}}{26}.$$

(i) Since we've determined that $\frac{1}{2}\beta$ is in the second quadrant and cosine is negative there, we have

$$\cos\left(\frac{\beta}{2}\right) = -\sqrt{\frac{1+\cos(\beta)}{2}} = -\sqrt{\frac{1+\frac{12}{13}}{2}} = -\sqrt{\frac{25}{26}} = -\frac{5\sqrt{26}}{26}.$$

3. (a) We have found that $\sin(\alpha+\beta) < 0$, which places the angle in either the third or fourth quadrant. Also, we've determined that $\cos(\alpha+\beta) < 0$, which means it must lie in either the second or third quadrant. Hence $\alpha+\beta$ is in the third quadrant.

   (b) We also deduced that $\sin(\alpha-\beta) < 0$ and $\cos(\alpha-\beta) < 0$, so using the same reasoning as in part (a), $\alpha-\beta$ must be in the third quadrant as well.

4. (a) We begin on the left:

$$\begin{aligned}\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} &= \frac{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)}\\ &= \frac{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)}\cdot\frac{\frac{1}{\cos(\alpha)\cos(\beta)}}{\frac{1}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{1-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{1+\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{1-\tan(\alpha)\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}.\end{aligned}$$

   (b) Again, we begin on the left:

$$\begin{aligned}\cos^2\left(\frac{x}{2}\right) &= \frac{1+\cos(x)}{2}\\ &= \frac{1+\cos(x)}{2}\cdot\frac{\sec(x)}{\sec(x)}\\ &= \frac{\sec(x)+1}{2\sec(x)}.\end{aligned}$$

   (c) We start on the left one more time:

$$\begin{aligned}\sin(2t)-\tan(t) &= 2\sin(t)\cos(t)-\frac{\sin(t)}{\cos(t)}\\ &= \frac{2\sin(t)\cos^2(t)-\sin(t)}{\cos(t)}\\ &= \frac{\sin(t)[2\cos^2(t)-1]}{\cos(t)}\\ &= \tan(t)\cos(2t).\end{aligned}$$

5. (a) Since the only other trig function in the equation is $\sin(x)$, we use the identity $\cos(2x) = 1-\sin^2(x)$, giving

$$\begin{aligned}\sin(x) &= 1-[1-2\sin^2(x)]\\ \sin(x) &= 2\sin^2(x)\\ 2\sin^2(x)-\sin(x) &= 0\\ \sin(x)[2\sin(x)-1] &= 0.\end{aligned}$$

If $\sin(x) = 0$ then either $x = 0$ or $x = \pi$. If $\sin(x) = \frac{1}{2}$ then $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Hence the three solutions are

$$x = 0, \quad \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \pi.$$

(b) We have

$$\begin{aligned} 2\sin(x)\cos(x) + 2\cos^2(x) &= 0 \\ 2\cos(x)[\sin(x) + \cos(x)] &= 0 \end{aligned}$$

so one possibility is that $\cos(x) = 0$, giving $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. The other possibility is that $\sin(x) = -\cos(x)$. Dividing both sides by $\cos(x)$, this becomes $\tan(x) = -1$, and so either $x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$. Hence the four solutions are

$$x = \frac{\pi}{2}, \quad \frac{3\pi}{4}, \quad \frac{3\pi}{2}, \quad \frac{7\pi}{4}.$$