

Section 1.6: Radicals

Def'n: If b is a real number and n is a natural number then an n th root of b is a real number c for which $c^n = b$.

Here, n is the index.

When $n=2$ we have a square root.

When $n=3$ we have a cube root.

Many numbers have a unique n th root which may be positive or negative.

eg Only $0^2 = 0$ so 0 is the only square root of 0 .

eg Only $(-4)^3 = -64$ so -4 is the only cube root of -64 .

However, some numbers have no n th roots. This occurs when b is negative and n is even.

eg There is no square root of -64 .

Furthermore, some numbers have two n th roots, one positive and one negative. This occurs when b is positive and n is even.

eg Both 8 and -8 are square roots of 64 because $8^2 = 64$ and $(-8)^2 = 64$.

Def'n: The principal nth root of a number b is its n th root if n is odd or $b=0$. It is the positive n th root if n is even and $b>0$. We denote the principal n th root as $\sqrt[n]{b}$. We call this a radical.

If $n=2$, we usually write $\sqrt{b}$ rather than $\sqrt[2]{b}$.

eg $\sqrt{64} = 8$

$$\sqrt{0} = 0$$

$$\sqrt[3]{-64} = -4$$

$\sqrt{-64}$ is undefined

If $c = \sqrt[n]{b}$ then we know that $c^n = b$. But then

$$(\sqrt[n]{b})^n = b.$$

Now suppose that $\sqrt[n]{b} = b^p$. Then we have

$$(b^p)^n = b$$

$$b^{p \cdot n} = b^1$$

so $p \cdot n = 1$ so $p = 1/n$. Therefore

$$\sqrt[n]{b} = b^{1/n}.$$

eg Since $\sqrt{64} = 8$ so we can also write $64^{1/2} = 8$.

Since $\sqrt[3]{-64} = -4$ so $(-64)^{1/3} = -4$.

Furthermore, $(\sqrt[n]{b})^m = (b^{1/n})^m = b^{\frac{1}{n} \cdot m} = b^{m/n}$.

eg $9^{3/2} = (9^{1/2})^3 = (\sqrt{9})^3 = 3^3 = \boxed{27}$

eg $(-8)^{5/3} = [(-8)^{1/3}]^5 = (\sqrt[3]{-8})^5 = (-2)^5 = \boxed{-32}$

eg $16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{(16^{1/4})^3} = \frac{1}{(\sqrt[4]{16})^3}$

$$= \frac{1}{2^3} = \boxed{\frac{1}{8}}$$

Properties of radicals

$$\textcircled{1} \quad \sqrt[n]{b} \cdot \sqrt[n]{c} = \sqrt[n]{bc}$$

This must be the case because $\sqrt[n]{b} = b^{1/n}$ and

$$\sqrt[n]{c} = c^{1/n} \text{ so}$$

$$\sqrt[n]{b} \cdot \sqrt[n]{c} = b^{1/n} \cdot c^{1/n}$$

$$= (bc)^{1/n}$$

$$= \sqrt[n]{bc}.$$

$$\text{eg } \sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} \boxed{= 4}$$

$$\textcircled{2} \quad \frac{\sqrt[n]{b}}{\sqrt[n]{c}} = \sqrt[n]{\frac{b}{c}}$$

$$\text{eg } \frac{\sqrt[3]{x}}{\sqrt[3]{x^5}} = \sqrt[3]{\frac{x}{x^5}} = \sqrt[3]{\frac{1}{x^4}} = \frac{\sqrt[3]{1}}{\sqrt[3]{x^4}} \boxed{= \frac{1}{\sqrt[3]{x^4}}}$$

$$\textcircled{3} \quad \sqrt[n]{\sqrt[m]{b}} = \sqrt[mn]{b}$$

$$\text{eg } \sqrt{\sqrt[3]{x}} = \sqrt[2]{\sqrt[3]{x}} = \sqrt[2 \cdot 3]{x} \boxed{= \sqrt[6]{x}}$$

It is often easier to manipulate radicals by writing them as exponents.

$$\begin{aligned} \text{eg } \sqrt{x} \cdot \sqrt[3]{x} &= x^{1/2} \cdot x^{1/3} = x^{1/2+1/3} \\ &= x^{3/6+2/6} \boxed{= x^{5/6}} = (\sqrt[6]{x})^5 \end{aligned}$$

$$\text{eg } \frac{6\sqrt[3]{x^4}}{2x^2} = \frac{6x^{4/3}}{2x^2} = 3x^{4/3-2} = 3x^{4/3-6/3}$$

$$= 3x^{-2/3} = \frac{3}{x^{2/3}}$$

We can extend our idea of factoring to include expressions involving radicals.

$$\text{eg Factor } 4x^{7/2} + 8x^{5/2} + 4x^{3/2}$$

We extract the largest common factor for which a polynomial will be left:

$$4x^{3/2}(x^2 + 2x + 1)$$

$$= 4x^{3/2}(x+1)(x+1)$$

$$= 4x^{3/2}(x+1)^2$$

We want to simplify radicals so that the argument of the radical (the expression inside the root) is reduced to the smallest possible terms.

$$\text{eg } \sqrt{32} = \sqrt{4 \cdot 4 \cdot 2} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$$

eg $\sqrt{x^2} \neq x$

In fact, $\sqrt{x^2} = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$

$$= |x|.$$

eg Assuming $x \geq 0$, simplify $\sqrt{50x^5} + \sqrt{18x^3}$.

$$\begin{aligned} \text{Here, } \sqrt{50x^5} &= \sqrt{25x^4 \cdot 2x} = \sqrt{25x^4} \cdot \sqrt{2x} \\ &= 5x^2 \sqrt{2x} \end{aligned}$$

$$\begin{aligned} \sqrt{18x^3} &= \sqrt{9x^2 \cdot 2x} = \sqrt{9x^2} \cdot \sqrt{2x} \\ &= 3x \sqrt{2x} \end{aligned}$$

$$\begin{aligned} \text{Thus } \sqrt{50x^5} + \sqrt{18x^3} &= 5x^2 \sqrt{2x} + 3x \sqrt{2x} \\ &= x \sqrt{2x} (5x + 3) \end{aligned}$$

$$= x \sqrt{2} \cdot \sqrt{x} (5x + 3)$$

$$\boxed{= \sqrt{2} x^{3/2} (5x + 3)}.$$

To remove a constant radical from the denominator of an expression, we rationalise the denominator.

First assume that the denominator consists of a single term of the form $k\sqrt{b}$. Then we multiply both the numerator and the denominator by $\sqrt{b}$.

eg Rationalise $\frac{9}{2\sqrt{3}}$.

$$\text{We write } \frac{9}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{2 \cdot 3}$$

$$= \frac{3\sqrt{3}}{2}$$

If the denominator of the expression is of the form

$$k\sqrt{b} + l\sqrt{c}$$

then we multiply the numerator and the denominator by the conjugate of the denominator, which has the form

$$k\sqrt{b} - l\sqrt{c}.$$

Here we can use the special product

$$(B+C)(B-C) = B^2 - C^2$$

Thus the denominator becomes

$$\begin{aligned} & (k\sqrt{b} + l\sqrt{c})(k\sqrt{b} - l\sqrt{c}) \\ &= (k\sqrt{b})^2 - (l\sqrt{c})^2 \\ &= k^2b - l^2c \end{aligned}$$

$$\text{eg } \frac{6}{3\sqrt{5} + 2\sqrt{3}} = \frac{6}{3\sqrt{5} + 2\sqrt{3}} \cdot \frac{3\sqrt{5} - 2\sqrt{3}}{3\sqrt{5} - 2\sqrt{3}}$$

$$= \frac{6(3\sqrt{5} - 2\sqrt{3})}{(3\sqrt{5})^2 - (2\sqrt{3})^2}$$

$$= \frac{6(3\sqrt{5} - 2\sqrt{3})}{9 \cdot 5 - 4 \cdot 3}$$

$$= \frac{6(3\sqrt{5} - 2\sqrt{3})}{33}$$

$$= \frac{2(3\sqrt{5} - 2\sqrt{3})}{11}$$

$$\text{eg } \frac{2}{3-4\sqrt{x}} = \frac{2}{3-4\sqrt{x}} \cdot \frac{3+4\sqrt{x}}{3+4\sqrt{x}}$$

$$= \frac{2(3+4\sqrt{x})}{3^2 - (4\sqrt{x})^2}$$

$$= \frac{2(3+4\sqrt{x})}{9-16x}$$