

Section 1.2: Exponents

Consider an expression of the form b^p . The real number b is the base and the real number p is the exponent (or the power).

If p is a natural number, we define

$$b^p = \underbrace{b \cdot b \cdot b \cdot b \cdots b}_{p \text{ factors}}$$

eg $3^1 = 3$

In fact, $b^1 = b$ for any b

$$3^2 = 3 \cdot 3 = 9 \text{ ("squared")}$$

$$3^3 = 3 \cdot 3 \cdot 3 = 27 \text{ ("cubed")}$$

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

$$\left(-\frac{1}{2}\right)^3 = \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) = -\frac{1}{8}$$

$$0^2 = 0 \cdot 0 = 0$$

Furthermore, $b^0 = 1$ for $b \neq 0$. However, 0^0 is undefined.

eg $3^0 = 1$

$$\left(-\frac{1}{2}\right)^0 = 1$$

We also define $b^{-p} = \frac{1}{b^p}$ for $b \neq 0$.

eg $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$$\left(-\frac{1}{2}\right)^{-3} = \frac{1}{\left(-\frac{1}{2}\right)^3} = \frac{1}{-1/8} = -8$$

Properties of exponents

$$\textcircled{1} b^p \cdot b^q = b^{p+q}$$

eg $(-2)^4 \cdot (-2)^5 = (-2)^{4+5} = (-2)^9 = -512$

$$\textcircled{2} \frac{b^p}{b^q} = b^{p-q}$$

eg $\frac{3^4}{3^7} = 3^{4-7} = 3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

eg $\frac{x^9}{x^5} = x^{9-5} = x^4$

$$\textcircled{3} (bc)^p = b^p c^p$$

eg $(5x)^2 = 5^2 x^2 = 25x^2$

Likewise, we can write $(bcd)^p = b^p c^p d^p$, and so on.

$$\textcircled{4} (b^p)^q = b^{pq}$$

eg $(3^2)^4 = 3^{2 \cdot 4} = 3^8 = 6561$

eg $(x^3)^{-2} = x^{3 \cdot (-2)} = x^{-6} = \frac{1}{x^6}$

$$\textcircled{5} \quad \left(\frac{b}{c}\right)^p = \frac{b^p}{c^p}$$

$$\text{eg} \quad \left(\frac{x^2}{y^3}\right)^4 = \frac{(x^2)^4}{(y^3)^4} = \frac{x^{2 \cdot 4}}{y^{3 \cdot 4}} = \frac{x^8}{y^{12}}$$

We can use Property $\textcircled{5}$ to observe that

$$\left(\frac{b}{c}\right)^{-1} = \frac{b^{-1}}{c^{-1}} = \frac{1/b}{1/c} = \frac{1}{b} \cdot \frac{c}{1} = \frac{c}{b}$$

Thus a fraction raised to a power of -1 is equivalent to taking the reciprocal of that fraction.

$$\text{More generally,} \quad \left(\frac{b}{c}\right)^{-p} = \left(\frac{c}{b}\right)^p = \frac{c^p}{b^p}$$

$$\begin{aligned} \text{eg} \quad \left(\frac{3x^4}{y}\right)^{-2} &= \left(\frac{y}{3x^4}\right)^2 \\ &= \frac{y^2}{(3x^4)^2} \\ &= \frac{y^2}{3^2(x^4)^2} \\ &= \frac{y^2}{9x^{4 \cdot 2}} \\ &= \frac{y^2}{9x^8} \end{aligned}$$

We want to look for combinations of the properties of exponents to help us simplify exponential expressions.

$$\begin{aligned}
 \text{eg } \left(\frac{(2x^3)^4}{8x^{-1}y^{-9}} \right)^2 &= \left(\frac{2^4(x^3)^4}{8x^{-1}y^{-9}} \right)^2 \\
 &= \left(\frac{16x^{12}}{8x^{-1}y^{-9}} \right)^2 \\
 &= \left(\frac{2x^{12}}{x^{-1}y^{-9}} \right)^2 \\
 &= \left(\frac{2x^{12-(-1)}}{y^{-9}} \right)^2 \\
 &= (2x^{13}y^9)^2 \\
 &= 2^2 x^{13 \cdot 2} y^{9 \cdot 2} \\
 &= 4x^{26}y^{18}
 \end{aligned}$$

While this kind of simplification can be achieved for expressions involving products and quotients, we must be careful when addition or subtraction is present.

$$\begin{aligned}
 \text{eg } \frac{1}{x^{-1}+y^{-1}} &\neq \frac{1}{x^{-1}} + \frac{1}{y^{-1}} = (x^{-1})^{-1} + (y^{-1})^{-1} \\
 &= x + y
 \end{aligned}$$

$$\begin{aligned}
 \text{In fact, } \frac{1}{x^{-1}+y^{-1}} &= \frac{1}{1/x + 1/y} \\
 &= \frac{1}{y/xy + x/xy} \\
 &= \frac{1}{(x+y)/xy} = \frac{xy}{x+y}
 \end{aligned}$$