MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 | **MATHEMATICS 1001** | WINTER 2025

SOLUTIONS

[6] 1. (a) Observe that

$$4x^3 + 4x^2 + 16x + 16 = 4x^2(x+1) + 16(x+1) = (4x+4)(x^2+4),$$

so the denominator has one unique linear factor and one unique irreducible quadratic factor. Hence the form of the partial fraction decomposition is

$$\frac{13x^2 - 4x - 12}{4x^3 + 4x^2 + 16x + 16} = \frac{A}{4x+4} + \frac{Bx+C}{x^2+4}$$
$$13x^2 - 4x - 12 = A(x^2+4) + (Bx+C)(4x+4).$$

When $x = -1$ we have $5 = 5A$ so $A = 1$. When $x = 0$, we have $-12 = 4A + 4C$ so $4C = -12 - 4A = -16$ and $C = -4$. Finally when, say, $x = 1$, we have $-3 = 5A + 8B + 8C$ so $8B = -3 - 5A - 8C = 24$ and $B = 3$. Thus

$$\begin{aligned}\int \frac{13x^2 - 4x - 12}{4x^3 + 4x^2 + 16x + 16}\,dx &= \int \left[\frac{\frac{1}{4}}{x+1} + \frac{3x-4}{x^2+4}\right] dx \\ &= \int \left[\frac{\frac{1}{4}}{x+1} + \frac{3x}{x^2+4} - \frac{4}{x^2+4}\right] dx \\ &= \frac{1}{4}\ln|x+1| - 2\arctan\left(\frac{x}{2}\right) + 3\int \frac{x}{x^2+4}\,dx.\end{aligned}$$

For the remaining integral, we let $u = x^2 + 4$ so $\frac{1}{2}\,du = x\,dx$ and we have

$$\begin{aligned}\int \frac{13x^2 - 4x - 12}{4x^3 + 4x^2 + 16x + 16}\,dx &= \frac{1}{4}\ln|x+1| - 2\arctan\left(\frac{x}{2}\right) + \frac{3}{2}\int \frac{1}{u}\,du \\ &= \frac{1}{4}\ln|x+1| - 2\arctan\left(\frac{x}{2}\right) + \frac{3}{2}\ln|u| + C \\ &= \frac{1}{4}\ln|x+1| - 2\arctan\left(\frac{x}{2}\right) + \frac{3}{2}\ln(x^2+4) + C.\end{aligned}$$

[5] (b) Since the power of cosine is odd, we set one factor aside and turn the remaining cosines into sines:

$$\begin{aligned}\int \sin^6(x)\cos^5(x)\,dx &= \int \sin^6(x)[\cos^2(x)]^2 \cdot \cos(x)\,dx \\ &= \int \sin^6(x)[1 - \sin^2(x)]^2 \cdot \cos(x)\,dx.\end{aligned}$$

Now we let $u = \sin(x)$ so $du = \cos(x)\,dx$. The integral becomes

$$\begin{aligned}\int \sin^6(x)\cos^5(x)\,dx &= \int u^6[1-u^2]^2\,du \\ &= \int u^6[1-2u^2+u^4]\,du \\ &= \int [u^6 - 2u^8 + u^{10}]\,du \\ &= \frac{u^7}{7} - 2\cdot\frac{u^9}{9} + \frac{u^{11}}{11} + C \\ &= \frac{1}{7}\sin^7(x) - \frac{2}{9}\sin^9(x) + \frac{1}{11}\sin^{11}(x) + C.\end{aligned}$$

[4] (c) The radical is in the form $\sqrt{x^2+k^2}$ with $k = 3$. We set $x = 3\tan(\theta)$ so $dx = 3\sec^2(\theta)\,d\theta$. Then

$$\sqrt{x^2+9} = \sqrt{9\tan^2(\theta)+9} = \sqrt{9\sec^2(\theta)} = 3\sec(\theta).$$

Now we have

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2+9}}\,dx &= \int \frac{1}{3\tan(\theta)\cdot 3\sec(\theta)}\cdot 3\sec^2(\theta)\,d\theta \\ &= \frac{1}{3}\int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta \\ &= \frac{1}{3}\int \frac{1}{\cos(\theta)}\cdot\frac{\cos(\theta)}{\sin(\theta)}\,d\theta \\ &= \frac{1}{3}\int \csc(\theta)\,d\theta \\ &= -\frac{1}{3}\ln|\csc(\theta)+\cot(\theta)| + C.\end{aligned}$$

Since we already know that $\sqrt{x^2+9} = 3\sec(\theta)$, we can construct a right triangle with interior angle θ, adjacent side of length 3 and hypotenuse of length $\sqrt{x^2+9}$. Then the opposite sidelength is x, and so

$$\csc(\theta) = \frac{\sqrt{x^2+9}}{x} \quad \text{and} \quad \cot(\theta) = \frac{3}{x}.$$

This means that

$$\int \frac{1}{x\sqrt{x^2+9}}\,dx = -\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}}{x} + \frac{3}{x}\right| + C = -\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}+3}{x}\right| + C.$$

[6] (d) Since

$$\int_1^{\sqrt{2}} \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\,dx = \int_1^{\sqrt{2}} \frac{x^2}{\left(\sqrt{4-x^2}\right)^3}\,dx,$$

the radical is in the form $\sqrt{k^2 - x^2}$ with $k = 2$. We let $x = 2\sin(\theta)$ so $dx = 2\cos(\theta)\,d\theta$, $x^2 = 4\sin^2(\theta)$ and

$$\sqrt{4 - x^2} = \sqrt{4 - 2\sin^2(\theta)} = \sqrt{4\cos^2(\theta)} = 2\cos(\theta).$$

When $x = 1$, $\sin(\theta) = \frac{1}{2}$ so $\theta = \frac{\pi}{6}$. When $x = \sqrt{2}$, $\sin(\theta) = \frac{\sqrt{2}}{2}$ so $\theta = \frac{\pi}{4}$. The integral becomes

$$\begin{aligned}
\int_1^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}}\,dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{4\sin^2(\theta)}{2\cos(\theta)} \cdot 2\cos(\theta)\,d\theta \\
&= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2(\theta)\,d\theta \\
&= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1 - \cos(2\theta)}{2}\,d\theta \\
&= 2\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
&= 2\left[\frac{\pi}{4} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right] \\
&= 2\left[\frac{\pi}{4} - \frac{1}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{4}\right] \\
&= \frac{\pi}{6} - 1 + \frac{\sqrt{3}}{2} \\
&= \frac{\pi - 6 + 3\sqrt{3}}{6}.
\end{aligned}$$

[5] 2. The integrand has a discontinuity at $x = 0$, the lower bound of integration. Thus we write

$$\int_0^2 \ln\left(\frac{x}{2}\right)\,dx = \lim_{T \to 0^+} \int_T^2 \ln\left(\frac{x}{2}\right)\,dx.$$

Now we use integration by parts with $w = \ln\left(\frac{x}{2}\right)$ so $dw = \frac{1}{x}\,dx$ and $dv = dx$ so $v = x$. The integral becomes

$$\begin{aligned}
\int_0^2 \ln\left(\frac{x}{2}\right)\,dx &= \lim_{T \to 0^+} \left(\left[x\ln\left(\frac{x}{2}\right)\right]_T^2 - \int_T^2 dx\right) \\
&= \lim_{T \to 0^+} \left[x\ln\left(\frac{x}{2}\right) - x\right]_T^2 \\
&= \lim_{T \to 0^+} \left[2\ln(1) - 2 - T\ln\left(\frac{T}{2}\right) + T\right] \\
&= -2 - \lim_{T \to 0^+} T\ln\left(\frac{T}{2}\right).
\end{aligned}$$

The remaining limit is a $0 \cdot \infty$ indeterminate form, so we first rewrite the limit as an $\frac{\infty}{\infty}$ form and apply l'Hôpital's Rule:

$$\begin{aligned}
\lim_{T\to 0^+} T \ln\left(\frac{T}{2}\right) &= \lim_{T\to 0^+} \frac{\ln\left(\frac{T}{2}\right)}{\frac{1}{T}} \\
&\overset{\text{H}}{=} \lim_{T\to 0^+} \frac{\frac{1}{T}}{-\frac{1}{T^2}} \\
&= \lim_{T\to 0^+} (-T) \\
&= 0.
\end{aligned}$$

Thus

$$\int_0^2 \ln\left(\frac{x}{2}\right)\, dx = -2 - 0 = -2.$$