MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 | **MATHEMATICS 1001** | WINTER 2023

SOLUTIONS

[3] 1. (a) We use integration by parts with $w = x$ so $dw = dx$, and $dv = \cos^2(x)\,dx$. Since

$$\int \cos^2(x)\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right] + C,$$

we have that $v = \frac{1}{2}x + \frac{1}{4}\sin(2x)$. Now

$$\begin{aligned}\int x\cos^2(x)\,dx &= x\left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right] - \int \left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right]dx \\ &= \frac{1}{2}x^2 + \frac{1}{4}x\sin(2x) - \frac{1}{2}\left(\frac{1}{2}x^2\right) - \frac{1}{4}\left[-\frac{1}{2}\cos(2x)\right] + C \\ &= \frac{1}{4}x^2 + \frac{1}{4}x\sin(2x) + \frac{1}{8}\cos(2x) + C.\end{aligned}$$

[5] (b) First observe that

$$x^3 - 3x^2 + 4x - 12 = x^2(x-3) + 4(x-3) = (x-3)(x^2+4),$$

so the form of the partial fraction decomposition is

$$\frac{2x+3}{x^3-3x^2+4x-12} = \frac{A}{x-3} + \frac{B+C}{x^2+4}.$$

Multiplying both sides by the original denominator, we have

$$2x + 3 = A(x^2+4) + (Bx+C)(x-3).$$

When $x = 3$, we have

$$9 = 13A \quad\Longrightarrow\quad A = \frac{9}{13}.$$

When $x = 0$, we have

$$3 = 4A - 3C = \frac{36}{13} - 3C \quad\Longrightarrow\quad 3C = -\frac{3}{13} \quad\Longrightarrow\quad C = -\frac{1}{13}.$$

And when, say, $x = 1$, we have

$$5 = 5A - 2(B+C) = \frac{45}{13} - 2B + \frac{2}{13} \quad\Longrightarrow\quad 2B = -\frac{18}{13} \quad\Longrightarrow\quad B = -\frac{9}{13}.$$

Thus the integral becomes

$$\int \frac{2x+3}{x^3-3x^2+4x-12}\,dx = \int \left[\frac{\frac{9}{13}}{x-3} + \frac{-\frac{9}{13}x - \frac{1}{13}}{x^2+4}\right] dx$$
$$= \int \left[\frac{9}{13}\cdot\frac{1}{x-3} - \frac{9}{13}\cdot\frac{x}{x^2+4} - \frac{1}{13}\cdot\frac{1}{x^2+4}\right] dx$$
$$= \frac{9}{13}\ln|x-3| - \frac{1}{26}\arctan\left(\frac{x}{2}\right) - \frac{9}{13}\int \frac{x}{x^2+4}\,dx.$$

For the remaining integral, we let $u = x^2+4$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. Thus

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+4) + C$$

and so

$$\int \frac{2x+3}{x^3-3x^2+4x-12}\,dx = \frac{9}{13}\ln|x-3| - \frac{1}{26}\arctan\left(\frac{x}{2}\right) - \frac{9}{26}\ln(x^2+4) + C.$$

[5] (c) Since the power of $\sin(x)$ is odd, we set aside one factor of $\sin(x)$ for u-substitution and write

$$\int \frac{\sin^7(x)}{\sqrt{\cos(x)}}\,dx = \int \frac{\sin^6(x)}{\sqrt{\cos(x)}}\cdot\sin(x)\,dx = \int \frac{[1-\cos^2(x)]^3}{\sqrt{\cos(x)}}\cdot\sin(x)\,dx.$$

Let $u = \cos(x)$ so $du = -\sin(x)\,dx$ and $-dx = \sin(x)\,dx$. Then the integral becomes

$$\int \frac{\sin^7(x)}{\sqrt{\cos(x)}}\,dx = -\int \frac{[1-u^2]^3}{\sqrt{u}}\,du$$
$$= -\int \frac{1-3u^2+3u^4-u^6}{\sqrt{u}}\,du$$
$$= -\int \left[u^{-\frac{1}{2}} - 3u^{\frac{3}{2}} + 3u^{\frac{7}{2}} - u^{\frac{11}{2}}\right] du$$
$$= -\left[2\sqrt{u} - \frac{6}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{9}{2}} - \frac{2}{13}u^{\frac{13}{2}}\right] + C$$
$$= -2\sqrt{\cos(x)} + \frac{6}{5}[\cos(x)]^{\frac{5}{2}} - \frac{2}{3}[\cos(x)]^{\frac{9}{2}} + \frac{2}{13}[\cos(x)]^{\frac{13}{2}} + C.$$

[7] (d) Let $x = \sin(\theta)$ so $dx = \cos(\theta)\,d\theta$ and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta).$$

When $x = \frac{\sqrt{2}}{2}$, we have

$$\theta = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

When $x = \frac{\sqrt{3}}{2}$, we have

$$\theta = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

The integral becomes

$$\begin{aligned}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin^3(\theta)}{\cos(\theta)} \cdot \cos(\theta)\,d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3(\theta)\,d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} [1-\cos^2(\theta)]\sin(\theta)\,d\theta.\end{aligned}$$

Let $u = \cos(\theta)$ so $-\,du = \sin(\theta)\,d\theta$. When $\theta = \frac{\pi}{4}$, $u = \frac{\sqrt{2}}{2}$. When $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Thus

$$\begin{aligned}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx &= -\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} [1-u^2]\,du \\ &= -\left[u - \frac{1}{3}u^3\right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \\ &= \frac{5\sqrt{2}}{12} - \frac{11}{24}.\end{aligned}$$

Alternatively, we could immediately let $u = 1 - x^2$ so $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. When $x = \frac{\sqrt{2}}{2}$, $u = \frac{1}{2}$. When $x = \frac{\sqrt{3}}{2}$, $u = \frac{1}{4}$. Thus the integral becomes

$$\begin{aligned}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int_{\frac{1}{2}}^{\frac{1}{4}} \frac{1-u}{\sqrt{u}}\,du \\ &= \frac{1}{2}\int_{\frac{1}{4}}^{\frac{1}{2}} [u^{-\frac{1}{2}} - u^{\frac{1}{2}}]\,du \\ &= \frac{1}{2}\left[2\sqrt{u} - \frac{2}{3}u^{\frac{3}{2}}\right]_{\frac{1}{4}}^{\frac{1}{2}} \\ &= \frac{5\sqrt{2}}{12} - \frac{11}{24}.\end{aligned}$$