MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.2 | **Math 1001 Worksheet** | WINTER 2023

SOLUTIONS

1. (a) We have

$$\begin{aligned} t^2y^2\frac{dy}{dt} &= 1 \\ y^2\,dy &= t^{-2}\,dt \\ \int y^2\,dy &= \int t^{-2}\,dt \\ \frac{1}{3}y^3 &= -\frac{1}{t} + C \\ y^3 &= C - \frac{3}{t} \\ y &= \sqrt[3]{C - \frac{3}{t}}. \end{aligned}$$

Since $y(3) = 1$, we obtain

$$y(3) = \sqrt[3]{C-1} = 1 \quad \Longrightarrow \quad C - 1 = 1$$

so $C = 2$. Thus the particular solution is

$$\boxed{y = \sqrt[3]{2 - \frac{3}{t}}.}$$

(b) We have

$$\begin{aligned} t^2y^2\frac{dy}{dt} &= \sqrt{1-y^2} \\ \frac{y^2}{\sqrt{1-y^2}}\,dy &= t^{-2}\,dt \\ \int \frac{y^2}{\sqrt{1-y^2}}\,dy &= \int t^{-2}\,dt. \end{aligned}$$

The integral on the right is elementary:

$$\int t^{-2}\,dt = -\frac{1}{t} + C.$$

However, the integral on the left requires trigonometric substitution. We let $y = \sin(\theta)$ so $dy = \cos(\theta)\,d\theta$ and

$$\sqrt{1-y^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta).$$

Hence

$$\begin{aligned}
\int \frac{y^2}{\sqrt{1-y^2}}\,dy &= \int \frac{\sin^2(\theta)}{\cos(\theta)} \cdot \cos(\theta)\,d\theta \\
&= \int \sin^2(\theta)\,d\theta \\
&= \int \frac{1-\cos(2\theta)}{2}\,d\theta \\
&= \frac{1}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta) + C \\
&= \frac{1}{2}\arcsin(y) - \frac{1}{2}y\sqrt{1-y^2} + C.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2}\arcsin(y) - \frac{1}{2}y\sqrt{1-y^2} &= -\frac{1}{t} + C \\
\arcsin(y) - y\sqrt{1-y^2} &= -\frac{2}{t} + C.
\end{aligned}$$

Note that it is not feasible to write the general solution in an explicit form. Nonetheless, since $y(-2) = 0$, we have

$$\arcsin(0) - 0 = 1 + C \quad \Longrightarrow \quad C = -1.$$

Hence the particular solution is given implicitly by

$$\arcsin(y) - y\sqrt{1-y^2} = -\frac{2}{t} - 1.$$

(c) We can separate the variables by writing

$$\begin{aligned}
\frac{dy}{dt} - ty^2 - 4t &= 0 \\
\frac{dy}{dt} &= ty^2 + 4t \\
\frac{dy}{dt} &= t(y^2 + 4) \\
\frac{1}{y^2+4}\,dy &= t\,dt \\
\int \frac{1}{y^2+4}\,dy &= \int t\,dt \\
\frac{1}{2}\arctan\left(\frac{y}{2}\right) &= \frac{1}{2}t^2 + C.
\end{aligned}$$

Since $y(1) = 2$, we have

$$\frac{1}{2}\arctan(1) = \frac{1}{2} + C \quad \Longrightarrow \quad C = \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2} = \frac{\pi}{8} - \frac{1}{2}.$$

Thus we can write the particular solution as

$$\begin{aligned}
\frac{1}{2}\arctan\left(\frac{y}{2}\right) &= \frac{1}{2}t^2 + \frac{\pi}{8} - \frac{1}{2} \\
\arctan\left(\frac{y}{2}\right) &= t^2 + \frac{\pi}{4} - 1 \\
\frac{y}{2} &= \tan\left(t^2 + \frac{\pi}{4} - 1\right) \\
y &= 2\tan\left(t^2 + \frac{\pi}{4} - 1\right).
\end{aligned}$$

(d) We have

$$\begin{aligned}
y\frac{dy}{dt} - e^{t+y} &= 0 \\
y\frac{dy}{dt} &= e^t e^y \\
ye^{-y}\,dy &= e^t\,dt \\
\int ye^{-y}\,dy &= \int e^t\,dt.
\end{aligned}$$

The integral on the right is elementary:

$$\int e^t\,dt = e^t + C.$$

The integral on the left requires integration by parts, with $w = y$ so $dw = dy$ and $dv = e^{-y}\,dy$ so $v = -e^{-y}$. Thus

$$\int ye^{-y}\,dy = -ye^{-y} + \int e^{-y}\,dy$$
$$= -ye^{-y} - e^{-y} + C.$$

Now the general solution is given by

$$-ye^{-y} - e^{-y} = e^t + C.$$

Since $y(0) = 0$, we have

$$0 - 1 = 1 + C \quad\Longrightarrow\quad C = -2$$

and so the particular solution is given implicitly by

$$-ye^{-y} - e^{-y} = e^t - 2 \quad \text{or} \quad ye^{-y} + e^{-y} = 2 - e^t.$$

(e) We have

$$\cos(y)\frac{dy}{dt} + \csc(y) = 0$$
$$\cos(y)\frac{dy}{dt} = -\csc(y)$$
$$\sin(y)\cos(y)\,dy = -\,dt$$
$$\int \sin(y)\cos(y)\,dy = -\int dt.$$

The integral on the right is elementary:

$$-\int dt = -t + C.$$

The integral on the left requires u-substitution, with $u = \sin(y)$ so $du = \cos(y)\,dy$. Thus

$$\int \sin(y)\cos(y)\,dy = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(y) + C.$$

Then the general solution is given by

$$\frac{1}{2}\sin^2(y) = -t + C.$$

Since $y\left(-\frac{1}{8}\right) = \frac{\pi}{6}$, we have

$$\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 = \frac{1}{8} + C \quad\Longrightarrow\quad \frac{1}{8} = \frac{1}{8} + C \quad\Longrightarrow\quad C = 0.$$

Hence the particular solution is

$$\begin{aligned}\frac{1}{2}\sin^2(y) &= -t\\ \sin^2(y) &= -2t\\ \sin(y) &= \sqrt{-2t}\\ y &= \arcsin\left(\sqrt{-2t}\right).\end{aligned}$$

2. (a) Since

$$y(t) = y_0 e^{kt},$$

we know that

$$y(2) = y_0 e^{2k} = 50 \quad \text{and} \quad y(5) = y_0 e^{5k} = 150.$$

Dividing the second by the first gives

$$\frac{y_0 e^{5k}}{y_0 e^{2k}} = \frac{150}{50} \quad \Longrightarrow \quad e^{3k} = 3 \quad \Longrightarrow \quad k = \frac{1}{3}\ln(3).$$

Hence using $y(2) = 50$ we have

$$50 = y_0 e^{\frac{2}{3}\ln(3)} \quad \Longrightarrow \quad y_0 = 50e^{-\frac{2}{3}\ln(3)} = 50 \cdot 3^{-\frac{2}{3}} = \frac{50}{\sqrt[3]{9}} \approx 24.$$

So there were about 24 parakeets originally on the island.

(b) We now have

$$y(t) = \frac{50}{\sqrt[3]{9}} e^{\frac{1}{3}\ln(3)t}$$

so

$$y(12) = \frac{50}{\sqrt[3]{9}} e^{\frac{1}{3}\ln(3)\cdot 12} = \frac{50}{\sqrt[3]{9}} e^{4\ln(3)} = \frac{50}{\sqrt[3]{9}} \cdot 3^4 = \frac{4050}{\sqrt[3]{9}} \approx 1947.$$

After seven more years, there will be approximately 1947 parakeets on the island!