MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.1 **Math 1001 Worksheet** WINTER 2023

SOLUTIONS

1. (a) The partial fraction decomposition is

$$\frac{3x-2}{x^2-x} = \frac{3x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \quad \Longrightarrow \quad 3x-2 = A(x-1)+Bx.$$

When $x=0$, $-A=-2$ so $A=2$. When $x=1$, we get $B=1$. Thus the integral becomes

$$\int \frac{3x-2}{x^2-x}\,dx = \int \left[\frac{2}{x} + \frac{1}{x-1}\right] dx$$
$$= 2\ln|x| + \ln|x-1| + C.$$

(b) The partial fraction decomposition is

$$\frac{2-x}{x^2+7x+10} = \frac{2-x}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} \quad \Longrightarrow \quad 2-x = A(x+2)+B(x+5).$$

When $x=-5$, $-3A=7$ so $A=-\frac{7}{3}$. When $x=-2$, $3B=4$ so $B=\frac{4}{3}$. The integral becomes

$$\int \frac{2-x}{x^2+7x+10}\,dx = \int \left[\frac{-\frac{7}{3}}{x+5} + \frac{\frac{4}{3}}{x+2}\right] dx$$
$$= -\frac{7}{3}\ln|x+5| + \frac{4}{3}\ln|x+2| + C.$$

(c) The partial fraction decomposition is

$$\frac{6x^3+x^2+25x+4}{x^4+7x^2+12} = \frac{6x^3+x^2+25x+4}{(x^2+4)(x^2+3)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+3}$$
$$6x^3+x^2+25x+4 = (Ax+B)(x^2+3)+(Cx+D)(x^2+4).$$

Unfortunately, there is no simple way to solve for the constants A, B, C and D, so we'll set up a system of four equations in the four unknowns. When $x=0$, we have

$$3B+4D=4.$$

When $x=1$, we have

$$4A+4B+5C+5D=36.$$

When $x = -1$, we have

$$4B - 4A + 5D - 5C = -26.$$

When $x = 2$, we have

$$14A + 7B + 16C + 8D = 106.$$

Adding the second and third equations gives

$$8B + 10D = 10 \implies 4B + 5D = 5.$$

Since the first equation implies that

$$D = 1 - \frac{3}{4}B,$$

we have

$$4B + 5\left(1 - \frac{3}{4}B\right) = 5 \implies \frac{1}{4}B = 0$$

so $B = 0$ and hence $D = 1$. Now the last two equations become

$$-4A - 5C = -31 \implies A = \frac{31}{4} - \frac{5}{4}C$$

and

$$14A + 16C = 98 \implies 7\left(\frac{31}{4} - \frac{5}{4}C\right) + 8C = 49 \implies -\frac{3}{4}C = -\frac{21}{4}$$

so $C = 7$ and therefore $A = -1$. The integral becomes

$$\begin{aligned}
\int \frac{6x^3 + x^2 + 25x + 4}{x^4 + 7x^2 + 12}\,dx &= \int \left[\frac{-x}{x^2+4} + \frac{7x+1}{x^2+3}\right] dx \\
&= \int \left[-\frac{x}{x^2+4} + \frac{7x}{x^2+3} + \frac{1}{x^2+3}\right] dx \\
&= -\frac{1}{2}\ln(x^2+4) + \frac{7}{2}\ln(x^2+3) + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}x}{3}\right) + C.
\end{aligned}$$

Here, the first two integrals can be evaluated by u-substitution with $u = x^2 + 4$ and $u = x^2 + 3$, respectively.

(d) The partial fraction decomposition is

$$\frac{-2x^3 + 12x^2 + 162x}{x^4 - 81} = \frac{-2x^3 + 12x^2 + 162x}{(x+3)(x-3)(x^2+9)} = \frac{A}{x-3} + \frac{B}{x+3} + \frac{Cx+D}{x^2+9}$$

$$\begin{aligned}
-2x^3 + 12x^2 + 162x &= A(x+3)(x^2+9) + B(x-3)(x^2+9) \\
&\quad + (Cx+D)(x+3)(x-3).
\end{aligned}$$

When $x = 3$, $108A = 540$ so $A = 5$. When $x = -3$, $-108B = -324$ so $B = 3$. When $x = 0$, $27A - 27B - 9D = 54 - 9D = 0$ so $D = 6$. And when $x = 1$, $40A - 20B - 8C - 8D = 92 - 8C = 172$ so $C = -10$. The integral becomes

$$\begin{aligned}\int \frac{-2x^3 + 12x^2 + 162x}{x^4 - 81}\,dx &= \int \left[\frac{5}{x-3} + \frac{3}{x+3} + \frac{-10x+6}{x^2+9}\right] dx \\ &= \int \left[\frac{5}{x-3} + \frac{3}{x+3} - \frac{10x}{x^2+9} + \frac{6}{x^2+9}\right] dx \\ &= 5\ln|x-3| + 3\ln|x+3| - 5\ln(x^2+9) + 2\arctan\left(\frac{x}{3}\right) + C.\end{aligned}$$

Here, the third integral can be evaluated by u-substitution with $u = x^2 + 9$.

2. (a) We have

$$\begin{aligned}\frac{7 + 5x - 2x^2}{(x-3)^3} &= \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} \\ 7 + 5x - 2x^2 &= A(x-3)^2 + B(x-3) + C \\ &= Ax^2 - 6Ax + 9A + Bx - 3B + C \\ &= (9A - 3B + C) + (B - 6A)x + Ax^2.\end{aligned}$$

(b) Comparing the coefficients of x^2, we immediately have $A = -2$. Comparing the coefficients of x, we have $B - 6A = B + 12 = 5$ so $B = -7$. Finally, comparing the constant coefficients, we have $9A - 3B + C = 3 + C = 7$ so $C = 4$.

(c) By the method of partial fractions, we can write

$$\begin{aligned}\int \frac{7 + 5x - 2x^2}{(x-3)^3}\,dx &= \int \left[\frac{-2}{x-3} + \frac{-7}{(x-3)^2} + \frac{4}{(x-3)^3}\right] dx \\ &= -2\ln|x-3| + \frac{7}{x-3} - \frac{2}{(x-3)^2} + C.\end{aligned}$$