## MEMORIAL UNIVERSITY OF NEWFOUNDLAND DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 3.1

Math 1001 Worksheet

WINTER 2025

## SOLUTIONS

1. (a) The partial fraction decomposition is

$$\frac{3x-2}{x^2-x} = \frac{3x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \implies 3x-2 = A(x-1) + Bx.$$

When x = 0, -A = -2 so A = 2. When x = 1, we get B = 1. Thus the integral becomes

$$\int \frac{3x-2}{x^2-x} \, dx = \int \left[\frac{2}{x} + \frac{1}{x-1}\right] \, dx$$
$$= 2\ln|x| + \ln|x-1| + C.$$

(b) The partial fraction decomposition is

 $\frac{2-x}{x^2+7x+10} = \frac{2-x}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} \implies 2-x = A(x+2) + B(x+5).$ 

When x = -5, -3A = 7 so  $A = -\frac{7}{3}$ . When x = -2, 3B = 4 so  $B = \frac{4}{3}$ . The integral becomes

$$\int \frac{2-x}{x^2+7x+10} \, dx = \int \left[\frac{-\frac{7}{3}}{x+5} + \frac{\frac{4}{3}}{x+2}\right] \, dx$$
$$= -\frac{7}{3}\ln|x+5| + \frac{4}{3}\ln|x+2| + C$$

(c) The partial fraction decomposition is

$$\frac{6x^3 + x^2 + 25x + 4}{x^4 + 7x^2 + 12} = \frac{6x^3 + x^2 + 25x + 4}{(x^2 + 4)(x^2 + 3)} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{x^2 + 3}$$
$$6x^3 + x^2 + 25x + 4 = (Ax + B)(x^2 + 3) + (Cx + D)(x^2 + 4).$$

Unfortunately, there is no simple way to solve for the constants A, B, C and D, so we'll set up a system of four equations in the four unknowns. When x = 0, we have

$$3B + 4D = 4.$$

When x = 1, we have

$$4A + 4B + 5C + 5D = 36.$$

When x = -1, we have

$$4B - 4A + 5D - 5C = -26.$$

When x = 2, we have

$$14A + 7B + 16C + 8D = 106$$

Adding the second and third equations gives

$$8B + 10D = 10 \implies 4B + 5D = 5.$$

Since the first equation implies that

$$D = 1 - \frac{3}{4}B,$$

we have

$$4B + 5\left(1 - \frac{3}{4}B\right) = 5 \quad \Longrightarrow \quad \frac{1}{4}B = 0$$

so B = 0 and hence D = 1. Now the last two equations become

$$-4A - 5C = -31 \implies A = \frac{31}{4} - \frac{5}{4}C$$

and

$$14A + 16C = 98 \implies 7\left(\frac{31}{4} - \frac{5}{4}C\right) + 8C = 49 \implies -\frac{3}{4}C = -\frac{21}{4}$$

so C = 7 and therefore A = -1. The integral becomes

$$\int \frac{6x^3 + x^2 + 25x + 4}{x^4 + 7x^2 + 12} \, dx = \int \left[ \frac{-x}{x^2 + 4} + \frac{7x + 1}{x^2 + 3} \right] \, dx$$
$$= \int \left[ -\frac{x}{x^2 + 4} + \frac{7x}{x^2 + 3} + \frac{1}{x^2 + 3} \right] \, dx$$
$$= -\frac{1}{2} \ln(x^2 + 4) + \frac{7}{2} \ln(x^2 + 3) + \frac{\sqrt{3}}{3} \arctan\left(\frac{\sqrt{3}x}{3}\right) + C.$$

Here, the first two integrals can be evaluated by *u*-substitution with  $u = x^2 + 4$  and  $u = x^2 + 3$ , respectively.

(d) The partial fraction decomposition is

$$\frac{-2x^3 + 12x^2 + 162x}{x^4 - 81} = \frac{-2x^3 + 12x^2 + 162x}{(x+3)(x-3)(x^2+9)} = \frac{A}{x-3} + \frac{B}{x+3} + \frac{Cx+D}{x^2+9}$$
$$-2x^3 + 12x^2 + 162x = A(x+3)(x^2+9) + B(x-3)(x^2+9)$$
$$+ (Cx+D)(x+3)(x-3).$$

When x = 3, 108A = 540 so A = 5. When x = -3, -108B = -324 so B = 3. When x = 0, 27A - 27B - 9D = 54 - 9D = 0 so D = 6. And when x = 1, 40A - 20B - 8C - 8D = 92 - 8C = 172 so C = -10. The integral becomes

$$\int \frac{-2x^3 + 12x^2 + 162x}{x^4 - 81} dx = \int \left[\frac{5}{x - 3} + \frac{3}{x + 3} + \frac{-10x + 6}{x^2 + 9}\right] dx$$
$$= \int \left[\frac{5}{x - 3} + \frac{3}{x + 3} - \frac{10x}{x^2 + 9} + \frac{6}{x^2 + 9}\right] dx$$
$$= 5\ln|x - 3| + 3\ln|x + 3| - 5\ln(x^2 + 9) + 2\arctan\left(\frac{x}{3}\right) + C.$$

Here, the third integral can be evaluated by u-substitution with  $u = x^2 + 9$ .

2. (a) We have

$$\frac{7+5x-2x^2}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$$
$$7+5x-2x^2 = A(x-3)^2 + B(x-3) + C$$
$$= Ax^2 - 6Ax + 9A + Bx - 3B + C$$
$$= (9A - 3B + C) + (B - 6A)x + Ax^2.$$

- (b) Comparing the coefficients of  $x^2$ , we immediately have A = -2. Comparing the coefficients of x, we have B 6A = B + 12 = 5 so B = -7. Finally, comparing the constant coefficients, we have 9A 3B + C = 3 + C = 7 so C = 4.
- (c) By the method of partial fractions, we can write

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$$\int \frac{7+5x-2x^2}{(x-3)^3} dx = \int \left[\frac{-2}{x-3} + \frac{-7}{(x-3)^2} + \frac{4}{(x-3)^3}\right] dx$$
$$= -2\ln|x-3| + \frac{7}{x-3} - \frac{2}{(x-3)^2} + C.$$