TEST 2 | **MATHEMATICS 1001-003** | WINTER 2025

SOLUTIONS

[7] 1. (a) We use a regular partition with subintervals of width

$$\Delta x = \frac{3-(-1)}{n} = \frac{4}{n}.$$

We choose the sample point

$$x_i^* = x_i = -1 + i\Delta x = -1 + \frac{4i}{n} = \frac{4i}{n} - 1.$$

Thus

$$\begin{aligned} f(x_i^*) &= 3 + 2\left(\frac{4i}{n} - 1\right) - \left(\frac{4i}{n} - 1\right)^2 \\ &= 3 + \frac{8i}{n} - 2 - \frac{16i^2}{n^2} + \frac{8i}{n} - 1 \\ &= \frac{16i}{n} - \frac{16i^2}{n^2}. \end{aligned}$$

Now we can write

$$\begin{aligned} \int_{-1}^{3} (3 + 2x - x^2)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{16i}{n} - \frac{16i^2}{n^2}\right) \cdot \frac{4}{n} \\ &= \lim_{n\to\infty} \left[\frac{64}{n^2} \sum_{i=1}^{n} i - \frac{64}{n^3} \sum_{i=1}^{n} i^2\right] \\ &= \lim_{n\to\infty} \left[\frac{64}{n^2} \cdot \frac{n(n+1)}{2} - \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\ &= 32 - \frac{64}{3} \\ &= \frac{32}{3}. \end{aligned}$$

[3] (b) We have

$$\begin{aligned} \int_{-1}^{3} (3 + 2x - x^2)\, dx &= \left[3x + 2 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{3} \\ &= (9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3}\right) \\ &= \frac{32}{3}. \end{aligned}$$

[5] 2. (a) We use integration by parts with $w = \ln(x)$ so $dw = \frac{1}{x}\,dx$ and $dv = \frac{1}{x^3}\,dx$ so $v = -\frac{1}{2x^2}$. Thus

$$\begin{aligned}\int_1^2 \frac{\ln(x)}{x^3}\,dx &= \left[-\frac{\ln(x)}{2x^2}\right]_1^2 + \frac{1}{2}\int_1^2 \frac{1}{x^3}\,dx \\ &= \left[-\frac{\ln(x)}{2x^2} - \frac{1}{4x^2}\right]_1^2 \\ &= -\frac{1}{8}\ln(2) - \frac{1}{16} + 0 + \frac{1}{4} \\ &= \frac{3}{16} - \frac{1}{8}\ln(2).\end{aligned}$$

[6] (b) Let $u = x^2$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. When $x = 0$, $u = 0$. When $x = \sqrt{2}$, $u = 2$. Thus the integral becomes

$$\begin{aligned}\int_0^{\sqrt{2}} \frac{x}{\sqrt{4-x^4}}\,dx &= \frac{1}{2}\int_0^2 \frac{1}{\sqrt{4-u^2}}\,du \\ &= \frac{1}{2}\left[\arcsin\left(\frac{u}{2}\right)\right]_0^2 \\ &= \frac{1}{2}\arcsin(1) - \frac{1}{2}\arcsin(0) \\ &= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot 0 \\ &= \frac{\pi}{4}.\end{aligned}$$

[4] (c) Observe that $2 - x = 0$ when $x = 2$, so

$$|2-x| = \begin{cases} 2-x, & \text{for } x \le 2 \\ -(2-x), & \text{for } x > 2. \end{cases}$$

Thus we can write

$$\begin{aligned}\int_{-3}^3 |2-x|\,dx &= \int_{-3}^2 |2-x|\,dx + \int_2^3 |2-x|\,dx \\ &= \int_{-3}^2 (2-x)\,dx - \int_2^3 (2-x)\,dx \\ &= \left[2x - \frac{x^2}{2}\right]_{-3}^2 - \left[2x - \frac{x^2}{2}\right]_2^3 \\ &= \left(4 - 2 + 6 + \frac{9}{2}\right) - \left(6 - \frac{9}{2} - 4 + 2\right) \\ &= 13.\end{aligned}$$

[5] 3. First we write

$$g(x) = \int_x^0 t\sqrt{t^3+1}\,dt + \int_0^{\sin(x)} t\sqrt{t^3+1}\,dt$$
$$= -\int_0^x t\sqrt{t^3+1}\,dt + \int_0^{\sin(x)} t\sqrt{t^3+1}\,dt.$$

Then we can use the First Fundamental Theorem of Calculus to obtain

$$g'(x) = -x\sqrt{x^3+1} + \sin(x)\sqrt{\sin^3(x)+1}\cdot[\sin(x)]'$$
$$= \sin(x)\cos(x)\sqrt{\sin^3(x)+1} - x\sqrt{x^3+1}.$$

[10] 4. (a) The sketch of R is given in Figure 1. Note that the two curves intersect when

$$\frac{1}{2}x^2 = 4\sqrt{x}$$
$$\frac{1}{4}x^4 = 16x$$
$$x^4 - 64x = 0$$
$$x(x^3-64) = 0,$$

that is, when $x = 0$ or $x = \sqrt[3]{64} = 4$. Substituting these values into either function shows that $y = 0$ and $y = 8$, respectively, so the points of intersection are $(0,0)$ and $(4,8)$.

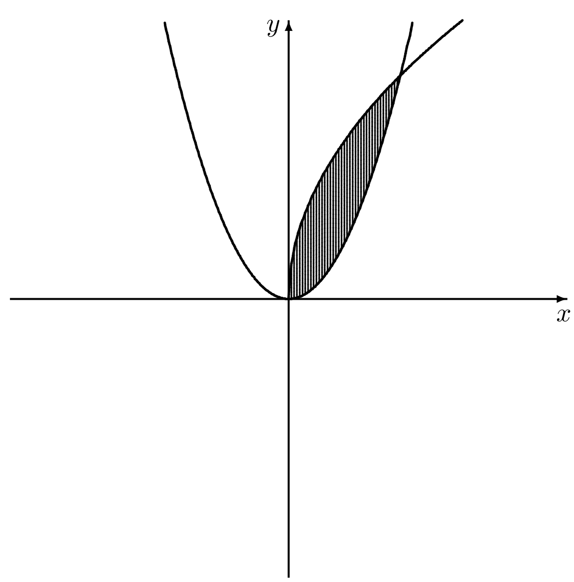

Figure 1: Question 4(a)

(b) The region is vertically simple. From the graph, we can see that the curve $f(x) = 4\sqrt{x}$ is the top boundary curve, while $g(x) = \frac{1}{2}x^2$ is the bottom boundary curve. Thus

$$A = \int_0^4 \left(4\sqrt{x} - \frac{1}{2}x^2\right)\,dx.$$

(c) The region is also horizontally simple. The function $y = \frac{1}{2}x^2$ can be written $x = \sqrt{2y}$ (since the square root is only defined for positive values of x), and thus $f(y) = \sqrt{2y}$ is the rightmost boundary curve. The function $y = 4\sqrt{x}$ can be written $x = \frac{1}{16}y^2$, and thus $g(y) = \frac{1}{16}y^2$ is the leftmost boundary curve. Hence

$$A = \int_0^8 \left(\sqrt{2y} - \frac{1}{16}y^2\right)\,dy.$$