

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2

MATHEMATICS 1001-001

WINTER 2025

SOLUTIONS

- [7] 1. (a) We use a regular partition with subintervals of width

$$\Delta x = \frac{4 - (-1)}{n} = \frac{5}{n}.$$

We choose the sample point

$$x_i^* = x_i = -1 + i\Delta x = -1 + \frac{5i}{n} = \frac{5i}{n} - 1.$$

Thus

$$\begin{aligned} f(x_i^*) &= 4 + 3\left(\frac{5i}{n} - 1\right) - \left(\frac{5i}{n} - 1\right)^2 \\ &= 4 + \frac{15i}{n} - 3 - \frac{25i^2}{n^2} + \frac{10i}{n} - 1 \\ &= \frac{25i}{n} - \frac{25i^2}{n^2}. \end{aligned}$$

Now we can write

$$\begin{aligned} \int_{-1}^4 (4 + 3x - x^2) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{25i}{n} - \frac{25i^2}{n^2} \right) \cdot \frac{5}{n} \\ &= \lim_{n \rightarrow \infty} \left[\frac{125}{n^2} \sum_{i=1}^n i - \frac{125}{n^3} \sum_{i=1}^n i^2 \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{125}{n^2} \cdot \frac{n(n+1)}{2} - \frac{125}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right] \\ &= \frac{125}{2} - \frac{125}{3} \\ &= \frac{125}{6}. \end{aligned}$$

- [3] (b) We have

$$\begin{aligned} \int_{-1}^4 (4 + 3x - x^2) dx &= \left[4x + 3 \cdot \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^4 \\ &= \left(16 + 24 - \frac{64}{3} \right) - \left(-4 + \frac{3}{2} + \frac{1}{3} \right) \\ &= \frac{125}{6}. \end{aligned}$$

- [5] 2. (a) We use integration by parts with $w = \ln(x)$ so $dw = \frac{1}{x} dx$ and $dv = \frac{1}{x^3} dx$ so $v = -\frac{1}{2x^2}$.
Thus

$$\begin{aligned}\int_1^2 \frac{\ln(x)}{x^3} dx &= \left[-\frac{\ln(x)}{2x^2} \right]_1^2 + \frac{1}{2} \int_1^2 \frac{1}{x^3} dx \\ &= \left[-\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} \right]_1^2 \\ &= -\frac{1}{8} \ln(2) - \frac{1}{16} + 0 + \frac{1}{4} \\ &= \frac{3}{16} - \frac{1}{8} \ln(2).\end{aligned}$$

- [6] (b) Let $u = x^2$ so $du = 2x dx$ and $\frac{1}{2} du = x dx$. When $x = 0$, $u = 0$. When $x = \sqrt{2}$, $u = 2$.
Thus the integral becomes

$$\begin{aligned}\int_0^{\sqrt{2}} \frac{x}{\sqrt{4-x^4}} dx &= \frac{1}{2} \int_0^2 \frac{1}{\sqrt{4-u^2}} du \\ &= \frac{1}{2} \left[\arcsin\left(\frac{u}{2}\right) \right]_0^2 \\ &= \frac{1}{2} \arcsin(1) - \frac{1}{2} \arcsin(0) \\ &= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot 0 \\ &= \frac{\pi}{4}.\end{aligned}$$

- [4] (c) Observe that $2 - x = 0$ when $x = 2$, so

$$|2 - x| = \begin{cases} 2 - x, & \text{for } x \leq 2 \\ -(2 - x), & \text{for } x > 2. \end{cases}$$

Thus we can write

$$\begin{aligned}\int_{-3}^3 |2 - x| dx &= \int_{-3}^2 |2 - x| dx + \int_2^3 |2 - x| dx \\ &= \int_{-3}^2 (2 - x) dx - \int_2^3 (2 - x) dx \\ &= \left[2x - \frac{x^2}{2} \right]_{-3}^2 - \left[2x - \frac{x^2}{2} \right]_2^3 \\ &= \left(4 - 2 + 6 + \frac{9}{2} \right) - \left(6 - \frac{9}{2} - 4 + 2 \right) \\ &= 13.\end{aligned}$$

[5] 3. First we write

$$\begin{aligned} g(x) &= \int_x^0 t\sqrt{t^3+1} dt + \int_0^{\sin(x)} t\sqrt{t^3+1} dt \\ &= -\int_0^x t\sqrt{t^3+1} dt + \int_0^{\sin(x)} t\sqrt{t^3+1} dt. \end{aligned}$$

Then we can use the First Fundamental Theorem of Calculus to obtain

$$\begin{aligned} g'(x) &= -x\sqrt{x^3+1} + \sin(x)\sqrt{\sin^3(x)+1} \cdot [\sin(x)]' \\ &= \sin(x)\cos(x)\sqrt{\sin^3(x)+1} - x\sqrt{x^3+1}. \end{aligned}$$

[10] 4. (a) The sketch of R is given in Figure 1. Note that the two curves intersect when

$$\begin{aligned} \frac{1}{2}x^2 &= 4\sqrt{x} \\ \frac{1}{4}x^4 &= 16x \\ x^4 - 64x &= 0 \\ x(x^3 - 64) &= 0, \end{aligned}$$

that is, when $x = 0$ or $x = \sqrt[3]{64} = 4$. Substituting these values into either function shows that $y = 0$ and $y = 8$, respectively, so the points of intersection are $(0, 0)$ and $(4, 8)$.

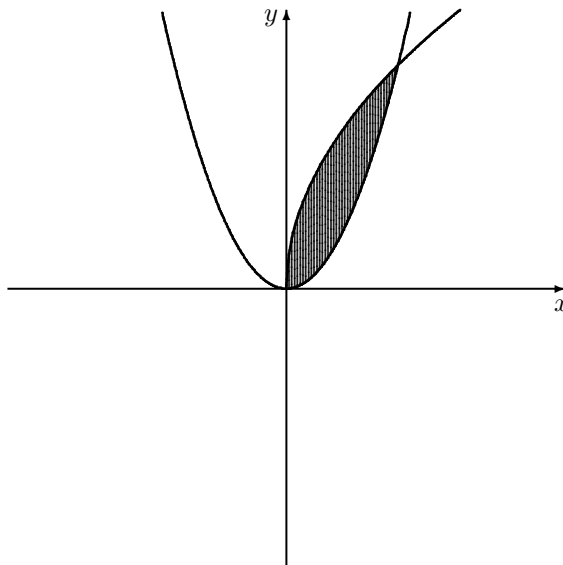


Figure 1: Question 4(a)

- (b) The region is vertically simple. From the graph, we can see that the curve $f(x) = 4\sqrt{x}$ is the top boundary curve, while $g(x) = \frac{1}{2}x^2$ is the bottom boundary curve. Thus

$$A = \int_0^4 \left(4\sqrt{x} - \frac{1}{2}x^2 \right) dx.$$

- (c) The region is also horizontally simple. The function $y = \frac{1}{2}x^2$ can be written $x = \sqrt{2y}$ (since the square root is only defined for positive values of x), and thus $f(y) = \sqrt{2y}$ is the rightmost boundary curve. The function $y = 4\sqrt{x}$ can be written $x = \frac{1}{16}y^2$, and thus $g(y) = \frac{1}{16}y^2$ is the leftmost boundary curve. Hence

$$A = \int_0^8 \left(\sqrt{2y} - \frac{1}{16}y^2 \right) dy.$$