

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.3

Math 1001 Worksheet

WINTER 2023

SOLUTIONS

1. (a) $F'(x) = (x^2 + 3)^{\cos(x)}$

(b) Let $u = \tan(x^2)$ so $F(u) = \int_0^u t dt$ and thus

$$F'(x) = F'(u)u' = u[2x \sec^2(x^2)] = 2x \tan(x^2) \sec^2(x^2).$$

(c) Let $u = e^x$ so

$$F(u) = \int_u^{100} e^t dt = - \int_{100}^u e^t dt.$$

Therefore

$$F'(x) = F'(u)u' = -e^u[e^x] = -e^x e^{e^x}.$$

(d) First observe that

$$\int_{x^3}^x \sqrt{t} dt = \int_{x^3}^0 \sqrt{t} dt + \int_0^x \sqrt{t} dt = - \int_0^{x^3} \sqrt{t} dt + \int_0^x \sqrt{t} dt.$$

We can differentiate the second integral using the Fundamental Theorem directly:

$$\frac{d}{dx} \left[\int_0^x \sqrt{t} dt \right] = \sqrt{x}.$$

For the first integral, let $u = x^3$ so then

$$\frac{d}{dx} \left[- \int_0^{x^3} \sqrt{t} dt \right] = \frac{d}{du} \left[- \int_0^u \sqrt{t} dt \right] \frac{du}{dx} = -\sqrt{u}[3x^2] = -3x^2\sqrt{x^3} = -3x^{\frac{7}{2}}.$$

Now we see that

$$F'(x) = -3x^{\frac{7}{2}} + \sqrt{x}.$$

2. (a) $\int_0^2 \frac{x^3}{4} dx = \frac{1}{4} \left[\frac{x^4}{4} \right]_0^2 = 1$

(b) $\int_2^3 (2 - 7x) dx = \int_2^3 2 dx - 7 \int_2^3 x dx = [2x]_2^3 - 7 \left[\frac{x^2}{2} \right]_2^3 = -\frac{31}{2}$

3. (a) This is an elementary integral:

$$\begin{aligned}\int_1^e (3x^{-3} + 5x^{-1} - 6x^2) dx &= \left[-\frac{3}{2}x^{-2} + 5\ln|x| - 2x^3 \right]_1^e \\ &= \left[-\frac{3}{2}e^{-2} + 5 - 2e^3 \right] - \left[-\frac{3}{2} + 0 - 2 \right] \\ &= \frac{17}{2} - \frac{3}{2e^2} - 2e^3.\end{aligned}$$

(b) We expand the product and integrate:

$$\begin{aligned}\int_0^1 (x+1)(2x-3) dx &= \int_0^1 (2x^2 - x - 3) dx \\ &= \left[\frac{2}{3}x^3 - \frac{1}{2}x^2 - 3x \right]_0^1 \\ &= -\frac{17}{6}.\end{aligned}$$

(c) Since this is the integral of a simple function with linear composition, we have

$$\begin{aligned}\int_{\frac{\pi}{8}}^{\pi} \cos(2x) dx &= \left[\frac{1}{2} \sin(2x) \right]_{\frac{\pi}{8}}^{\pi} \\ &= \frac{1}{2} \left[\sin(2\pi) - \sin\left(\frac{\pi}{4}\right) \right] \\ &= \frac{1}{2} \left[0 - \frac{\sqrt{2}}{2} \right] \\ &= -\frac{\sqrt{2}}{4}.\end{aligned}$$

(d) This is also the integral of a simple function with linear composition:

$$\begin{aligned}\int_2^0 (4t+1)^{-\frac{5}{2}} dt &= \left[\frac{1}{4} \cdot \frac{(4t+1)^{-\frac{3}{2}}}{-\frac{2}{3}} \right]_2^0 \\ &= -\frac{1}{6} \left[1^{-\frac{3}{2}} - 9^{-\frac{3}{2}} \right] \\ &= -\frac{1}{6} \left[1 - \frac{1}{27} \right] \\ &= -\frac{13}{81}.\end{aligned}$$

- (e) The integrand is an improper rational function so we could proceed via long division. However, the similarity between the numerator and the denominator lets us take a short-cut:

$$\begin{aligned}
 \int_{-2}^0 \frac{3x+8}{3x+7} dx &= \int_{-2}^0 \frac{(3x+7)+1}{3x+7} dx \\
 &= \int_{-2}^0 \left(1 + \frac{1}{3x+7} \right) dx \\
 &= \left[x + \frac{1}{3} \ln|3x+7| \right]_{-2}^0 \\
 &= \left[0 + \frac{1}{3} \ln(7) \right] - \left[-2 + \frac{1}{3} \ln(1) \right] \\
 &= 2 + \frac{\ln(7)}{3}.
 \end{aligned}$$

- (f) We simplify the integrand using long division:

$$\begin{array}{r}
 \overline{) -x^3 + 3x^2 } \\
 \underline{-x^3 + x } \\
 3x^2 + x + 3 \\
 \underline{3x^2 + 3} \\
 x
 \end{array}$$

so

$$\frac{3 + 3x^2 - x^3}{x^2 + 1} = -x + 3 + \frac{x}{x^2 + 1}.$$

The integral becomes

$$\int \frac{3 + 3x^2 - x^3}{x^2 + 1} dx = \int_{-2}^2 (-x + 3) dx + \int_{-2}^2 \frac{x}{x^2 + 1} dx.$$

We can handle the first integral using elementary techniques, but we must use u -substitution to evaluate the second integral. Let $u = x^2 + 1$ so $\frac{1}{2} du = x dx$. When $x = -2$, $u = 5$. When $x = 2$, $u = 5$ as well. Hence the integral now becomes

$$\begin{aligned}
 \frac{3 + 3x^2 - x^3}{x^2 + 1} dx &= \int_{-2}^2 (-x + 3) dx + \frac{1}{2} \int_5^5 \frac{du}{u} \\
 &= \left[-\frac{1}{2}x^2 + 3x \right]_{-2}^2 + 0 \\
 &= 12.
 \end{aligned}$$

- (g) Let $u = \ln(x + 2)$ so $du = \frac{dx}{x + 2}$. When $x = -1$, $u = \ln(1) = 0$. When $x = e^3 - 2$, $u = \ln(e^3) = 3$. The integral becomes

$$\begin{aligned} \int_{-1}^{e^3-2} \frac{\ln(x+2)}{x+2} dx &= \int_0^3 u du \\ &= \left[\frac{1}{2} u^2 \right]_0^3 \\ &= \frac{9}{2}. \end{aligned}$$

- (h) Let $u = \cos(\theta)$ so $-du = \sin(\theta) d\theta$. When $x = 0$, $u = \cos(0) = 1$. When $x = \pi$, $u = \cos(\pi) = -1$. The integral becomes

$$\begin{aligned} \int_0^\pi \cos(\cos(\theta)) \sin(\theta) d\theta &= - \int_1^{-1} \cos(u) du \\ &= - \left[\sin(u) \right]_1^{-1} \\ &= \sin(1) - \sin(-1) \\ &= 2 \sin(1) \end{aligned}$$

where we have simplified our answer using the fact that $\sin(-x) = -\sin(x)$ for all x .

- (i) Let $u = 2 + \frac{1}{x}$ so $-du = \frac{dx}{x^2}$. When $x = 1$, $u = 3$. When $x = 4$, $u = \frac{9}{4}$. The integral becomes

$$\begin{aligned} \int_{\frac{1}{2}}^4 \frac{1}{x^2} \sqrt{2 + \frac{1}{x}} dx &= - \int_4^{\frac{9}{4}} \sqrt{u} du \\ &= - \left[\frac{2}{3} u^{\frac{3}{2}} \right]_4^{\frac{9}{4}} \\ &= \frac{37}{12}. \end{aligned}$$

- (j) Let $u = -\cot\left(\frac{x}{3}\right)$ so $3 du = \csc^2\left(\frac{x}{3}\right) dx$. When $x = \frac{\pi}{2}$, $u = -\cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$. When $x = 3\pi$, $u = -\cot\left(\frac{\pi}{2}\right) = 0$. The integral becomes

$$\begin{aligned} \int_{\frac{\pi}{2}}^{3\pi} \csc^2\left(\frac{x}{3}\right) \left[1 - e^{-\cot\left(\frac{x}{3}\right)}\right] dx &= 3 \int_{-\sqrt{3}}^0 (1 - e^u) du \\ &= 3 \left[u - e^u \right]_{-\sqrt{3}}^0 \\ &= 3\sqrt{3} - 3 + 3e^{-\sqrt{3}}. \end{aligned}$$

- (k) Let $u = k^3 - x^3$ so $-\frac{1}{3} du = x^2 dx$. When $x = 0$, $u = k^3$. When $x = k$, $u = 0$. The integral becomes

$$\begin{aligned} \int_0^k x^2(k^3 - x^3)^{\frac{4}{3}} dx &= -\frac{1}{3} \int_{k^3}^0 u^{\frac{4}{3}} du \\ &= -\frac{1}{3} \left[\frac{3}{7} u^{\frac{7}{3}} \right]_{k^3}^0 \\ &= \frac{1}{7} k^7. \end{aligned}$$

- (l) Note that we can write

$$\int_{\sqrt{3}}^{\sqrt{2}} \frac{1}{\sqrt{1 - \frac{x^2}{4}}} dx = \int_{\sqrt{3}}^{\sqrt{2}} \frac{1}{\sqrt{1^2 - \left(\frac{x}{2}\right)^2}} dx,$$

so we let $u = \frac{x}{2}$ and therefore $2 du = dx$. When $x = \sqrt{3}$, $u = \frac{\sqrt{3}}{2}$. When $x = \sqrt{2}$, $u = \frac{\sqrt{2}}{2}$. The integral becomes

$$\begin{aligned} \int_{\sqrt{3}}^{\sqrt{2}} \frac{1}{\sqrt{1 - \frac{x^2}{4}}} dx &= 2 \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1^2 - u^2}} du \\ &= 2 \left[\arcsin(u) \right]_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} \\ &= -\frac{\pi}{6}. \end{aligned}$$

- (m) Let $u = \sqrt{t}$ so $2 du = t^{-\frac{1}{2}} dt$ and $t = u^2$. When $t = 0$, $u = 0$. When $t = 4$, $u = 2$. The integral becomes

$$\begin{aligned} \int_0^4 \frac{1}{\sqrt{t}(t+4)} dt &= 2 \int_0^2 \frac{1}{u^2+2^2} du \\ &= 2 \left[\frac{1}{2} \arctan\left(\frac{u}{2}\right) \right]_0^2 \\ &= \frac{\pi}{4}. \end{aligned}$$

- (n) Note first that

$$\int_{\sqrt{e}}^e \frac{1}{x \ln(x) \sqrt{16(\ln(x))^2 - 4}} dx = \int_{\sqrt{e}}^e \frac{1}{x \ln(x) \sqrt{(4 \ln(x))^2 - 2^2}} dx.$$

Then let $u = 4 \ln(x)$ so $\frac{1}{4} du = \frac{dx}{x}$ and $\frac{1}{4}u = \ln(x)$. When $x = \sqrt{e}$, $u = 4 \ln(\sqrt{e}) = 4 \ln(e^{\frac{1}{2}}) = 2$. When $x = e$, $u = 4 \ln(e) = 4$. The integral becomes

$$\begin{aligned} \int_{\sqrt{e}}^e \frac{1}{x \ln(x) \sqrt{16(\ln(x))^2 - 4}} dx &= \frac{1}{4} \int_2^4 \frac{1}{\left(\frac{u}{4}\right) \sqrt{u^2 - 2^2}} du \\ &= \int_2^4 \frac{1}{u \sqrt{u^2 - 2^2}} du \\ &= \left[\frac{1}{2} \operatorname{arcsec} \left(\frac{u}{2} \right) \right]_2^4 \\ &= \frac{\pi}{6}. \end{aligned}$$

(o) We use integration by parts. Let $w = \ln(x)$ so $dw = x^{-1} dx$, and let $dv = x^{-4} dx$ so $v = -\frac{1}{3}x^{-3}$. Then

$$\begin{aligned} \int_1^3 \frac{\ln(x)}{x^4} dx &= \left[-\frac{\ln(x)}{3x^3} \right]_1^3 + \frac{1}{3} \int_1^3 x^{-4} dx \\ &= \left[-\frac{\ln(x)}{3x^3} - \frac{1}{9x^3} \right]_1^3 \\ &= \frac{26}{243} - \frac{\ln(3)}{81}. \end{aligned}$$

(p) Let $u = \sin(x)$ so $du = \cos(x) dx$. When $x = \arcsin\left(\frac{3}{5}\right)$, $u = \frac{3}{5}$. When $x = \frac{\pi}{2}$, $u = 1$. Thus the integral becomes

$$\int_{\arcsin\left(\frac{3}{5}\right)}^{\frac{\pi}{2}} \cos(x) \ln(\sin(x)) dx = \int_{\frac{3}{5}}^1 \ln(u) du.$$

Now we use integration by parts. Let $w = \ln(u)$ so $dw = \frac{1}{u} du$. Let $dv = du$ so $v = u$. Then

$$\begin{aligned} \int_{\arcsin\left(\frac{3}{5}\right)}^{\frac{\pi}{2}} \cos(x) \ln(\sin(x)) dx &= \left[u \ln(u) \right]_{\frac{3}{5}}^1 - \int_{\frac{3}{5}}^1 u \cdot \frac{1}{u} du \\ &= \left[u \ln(u) \right]_{\frac{3}{5}}^1 - \int_{\frac{3}{5}}^1 du \\ &= \left[u \ln(u) - u \right]_{\frac{3}{5}}^1 \\ &= -\frac{2}{5} - \frac{3}{5} \ln\left(\frac{3}{5}\right). \end{aligned}$$

- (q) First we make the substitution $u = x^2$ so $\frac{1}{2} du = x dx$. When $x = 0$, $u = 0$. When $x = 1$, $u = 1$. The integral becomes

$$\int_0^1 x \arcsin(x^2) dx = \frac{1}{2} \int_0^1 \arcsin(u) du.$$

Now we use integration by parts. Let $w = \arcsin(u)$ so $dw = \frac{1}{\sqrt{1-u^2}} du$, and let $dv = du$ so $v = u$. Then

$$\int_0^1 x \arcsin(x^2) dx = \frac{1}{2} \left[u \arcsin(u) \right]_0^1 - \frac{1}{2} \int_0^1 \frac{u}{\sqrt{1-u^2}} du.$$

Now we have to make another substitution: let $z = 1 - u^2$ so $-\frac{1}{2} dz = u du$. When $u = 0$, $z = 1$. When $u = 1$, $z = 0$. So then

$$\begin{aligned} \int_0^1 x \arcsin(x^2) dx &= \frac{1}{2} \left[u \arcsin(u) \right]_0^1 + \frac{1}{4} \int_1^0 z^{-\frac{1}{2}} dz \\ &= \left[\frac{1}{2} u \arcsin(u) \right]_0^1 + \frac{1}{2} \left[\sqrt{z} \right]_1^0 \\ &= \frac{\pi}{4} - \frac{1}{2}. \end{aligned}$$

- (r) First observe that we can write

$$|2x + 8| = \begin{cases} 2x + 8 & \text{for } x \geq -4 \\ -(2x + 8) & \text{for } x < -4. \end{cases}$$

Using the Additive Interval Property, then, we can write

$$\begin{aligned} \int_{-5}^{-1} |2x + 8| dx &= \int_{-5}^{-4} |2x + 8| dx + \int_{-4}^{-1} |2x + 8| dx \\ &= - \int_{-5}^{-4} (2x + 8) dx + \int_{-4}^{-1} (2x + 8) dx \\ &= - \left[x^2 + 8x \right]_{-5}^{-4} + \left[x^2 + 8x \right]_{-4}^{-1} \\ &= 10. \end{aligned}$$

- (s) Again, we begin by rewriting the absolute value as a piecewise function; because the function inside the absolute value is more complicated than those we have previously considered, we need to do a little more work to see how to write it in this form. We are interested in the intervals where $x^2 - 4x + 3$ is positive or negative, so we set

$$x^2 - 4x + 3 = (x - 3)(x - 1) = 0,$$

giving $x = 1$ or $x = 3$. This means that there are three intervals of interest: $x < 1$, $1 < x < 3$ and $x > 3$. The function $x^2 - 4x + 3$ is either positive for all x , or negative for all x , on each of these intervals. We just need to test a value of x in each interval to see which is which. For $x < 1$, try $x = 0$. Then

$$x^2 - 4x + 3 = 0^2 - 4(0) + 3 = 3,$$

so $x^2 - 4x + 3 > 0$ for $x < 1$ (which means that the absolute value has no effect). For $1 < x < 3$, try $x = 2$. Then

$$x^2 - 4x + 3 = 2^2 - 4(2) + 3 = -1,$$

so $x^2 - 4x + 3 < 0$ for $1 < x < 3$ and therefore

$$|x^2 - 4x + 3| = -(x^2 - 4x + 3)$$

on this interval. Finally, for $x > 3$, we try $x = 4$. Then

$$x^2 - 4x + 3 = 4^2 - 4(4) + 3 = 3,$$

so $x^2 - 4x + 3 > 0$ for $x > 3$ and, again, the absolute value does nothing. Hence the piecewise definition of the integrand is

$$|x^2 - 4x + 3| = \begin{cases} x^2 - 4x + 3 & \text{for } x \leq 1 \text{ and } x \geq 3 \\ -(x^2 - 4x + 3) & \text{for } 1 < x < 3. \end{cases}$$

Finally, using the Additive Interval Property, we obtain

$$\begin{aligned} & \int_{-1}^4 |x^2 - 4x + 3| dx \\ &= \int_{-1}^1 |x^2 - 4x + 3| dx + \int_1^3 |x^2 - 4x + 3| dx + \int_3^4 |x^2 - 4x + 3| dx \\ &= \int_{-1}^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx + \int_3^4 (x^2 - 4x + 3) dx \\ &= \left[\frac{1}{3}x^3 - 2x^2 + 3x \right]_{-1}^1 - \left[\frac{1}{3}x^3 - 2x^2 + 3x \right]_1^3 + \left[\frac{1}{3}x^3 - 2x^2 + 3x \right]_3^4 \\ &= \frac{28}{3}. \end{aligned}$$

$$4. \quad A = \int_{\frac{11}{4}}^{\frac{35}{4}} \frac{2}{\sqrt{2x - \frac{3}{2}}} dx = 2 \left[\frac{1}{2} \cdot \frac{\sqrt{2x - \frac{3}{2}}}{\frac{1}{2}} \right]_{\frac{11}{4}}^{\frac{35}{4}} = 2[4 - 2] = 4$$