

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.4

Math 1001 Worksheet

WINTER 2025

SOLUTIONS

1. (a) Let $w = x$ so $dw = dx$, and let $dv = \cos(x) dx$ so $v = \sin(x)$. Then

$$\int x \cos(x) dx = x \sin(x) - \int \sin(x) dx = x \sin(x) + \cos(x) + C.$$

- (b) Let $w = x^2$ so $dw = 2x dx$, and let $dv = \cos(x) dx$ so $v = \sin(x)$. Then

$$\int x^2 \cos(x) dx = x^2 \sin(x) - 2 \int x \sin(x) dx.$$

Now let $w = x$ so $dw = dx$ and let $dv = \sin(x) dx$ so $v = -\cos(x)$. Then we obtain

$$\begin{aligned} \int x^2 \cos(x) dx &= x^2 \sin(x) - 2 \left[-x \cos(x) + \int \cos(x) dx \right] \\ &= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C. \end{aligned}$$

- (c) Let $w = x$ so $dw = dx$, and let $dv = \tan(x) \sec(x) dx$ so $v = \sec(x)$. Then

$$\int x \tan(x) \sec(x) dx = x \sec(x) - \int \sec(x) dx = x \sec(x) - \ln|\sec(x) + \tan(x)| + C.$$

- (d) Let $w = y^4$ so $dw = 4y^3 dy$. Let $dv = y^3 e^{y^4} dy$. To integrate this to get v we must make a substitution; let $u = y^4$ so $\frac{1}{4} du = y^3 dy$ and thus

$$v = \int dv = \frac{1}{4} \int e^u du = \frac{1}{4} e^u = \frac{1}{4} e^{y^4}.$$

Then, returning to the original integral, we have

$$\int y^7 e^{y^4} dy = \int y^4 y^3 e^{y^4} dy = \frac{1}{4} y^4 e^{y^4} - \int y^3 e^{y^4} dy.$$

Using the same u -substitution as before, this becomes

$$\int y^7 e^{y^4} dy = \frac{1}{4} y^4 e^{y^4} - \frac{1}{4} \int e^u du = \frac{1}{4} y^4 e^{y^4} - \frac{1}{4} e^u + C = \frac{1}{4} y^4 e^{y^4} - \frac{1}{4} e^{y^4} + C.$$

Alternatively, we can make the substitution right away: let $u = y^4$ so $\frac{1}{4} du = y^3 dy$. Then

$$\int y^7 e^{y^4} dy = \int y^4 y^3 e^{y^4} dy = \frac{1}{4} \int u e^u du.$$

Now we use parts, setting $w = u$ so $dw = du$ and $dv = e^u dw$ so $v = e^u$. Then

$$\int y^7 e^{y^4} dy = \frac{1}{4} \left[u e^u - \int e^u du \right] = \frac{1}{4} u e^u - \frac{1}{4} e^u + C = \frac{1}{4} y^4 e^{y^4} - \frac{1}{4} e^{y^4} + C,$$

as before.

(e) Let $w = e^{3x}$ so $dw = 3e^{3x} dx$, and let $dv = \sin(5x) dx$ so $v = -\frac{1}{5} \cos(5x)$. Then

$$\int e^{3x} \sin(5x) dx = -\frac{1}{5} e^{3x} \cos(5x) + \frac{3}{5} \int e^{3x} \cos(5x) dx.$$

Again let $w = e^{3x}$ so $dw = 3e^{3x} dx$, and now let $dv = \cos(5x) dx$ so $v = \frac{1}{5} \sin(5x)$. This gives

$$\begin{aligned} \int e^{3x} \sin(5x) dx &= -\frac{1}{5} e^{3x} \cos(5x) + \frac{3}{5} \left[\frac{1}{5} e^{3x} \sin(5x) - \frac{3}{5} \int e^{3x} \sin(5x) dx \right] \\ &= -\frac{1}{5} e^{3x} \cos(5x) + \frac{3}{25} e^{3x} \sin(5x) - \frac{9}{25} \int e^{3x} \sin(5x) dx \\ \frac{34}{25} \int e^{3x} \sin(5x) dx &= -\frac{1}{5} e^{3x} \cos(5x) + \frac{3}{25} e^{3x} \sin(5x) \\ \int e^{3x} \sin(5x) dx &= -\frac{5}{34} e^{3x} \cos(5x) + \frac{3}{34} e^{3x} \sin(5x) + C. \end{aligned}$$

(f) Let $w = \cos\left(\frac{2x}{3}\right)$ so $dw = -\frac{2}{3} \sin\left(\frac{2x}{3}\right) dx$, and let $dv = \cos(x) dx$ so $v = \sin(x)$. Then

$$\int \cos(x) \cos\left(\frac{2x}{3}\right) dx = \sin(x) \cos\left(\frac{2x}{3}\right) + \frac{2}{3} \int \sin(x) \sin\left(\frac{2x}{3}\right) dx.$$

Now let $w = \sin\left(\frac{2x}{3}\right) dx$ so $dw = \frac{2}{3} \cos\left(\frac{2x}{3}\right) dx$, and let $dv = \sin(x) dx$ so $v = -\cos(x) dx$. We now obtain

$$\begin{aligned} \int \cos(x) \cos\left(\frac{2x}{3}\right) dx &= \sin(x) \cos\left(\frac{2x}{3}\right) \\ &\quad + \frac{2}{3} \left[-\cos(x) \sin\left(\frac{2x}{3}\right) + \frac{2}{3} \int \cos(x) \cos\left(\frac{2x}{3}\right) dx \right] \\ &= \sin(x) \cos\left(\frac{2x}{3}\right) - \frac{2}{3} \cos(x) \sin\left(\frac{2x}{3}\right) \\ &\quad + \frac{4}{9} \int \cos(x) \cos\left(\frac{2x}{3}\right) dx \\ \frac{5}{9} \int \cos(x) \cos\left(\frac{2x}{3}\right) dx &= \sin(x) \cos\left(\frac{2x}{3}\right) - \frac{2}{3} \cos(x) \sin\left(\frac{2x}{3}\right) \\ \int \cos(x) \cos\left(\frac{2x}{3}\right) dx &= \frac{9}{5} \sin(x) \cos\left(\frac{2x}{3}\right) - \frac{6}{5} \cos(x) \sin\left(\frac{2x}{3}\right) + C. \end{aligned}$$

(g) Let $w = \arcsin(6x)$ so $dw = \frac{6}{\sqrt{1-36x^2}}$. Let $dv = dx$ so $v = x$. Then

$$\int \arcsin(6x) dx = x \arcsin(6x) - 6 \int \frac{x}{\sqrt{1-36x^2}} dx.$$

Now we use u -substitution with $u = 1 - 36x^2$ so $-\frac{1}{72} du = x dx$. The integral becomes

$$\begin{aligned}\int \arcsin(6x) dx &= x \arcsin(6x) + 6 \left[\frac{1}{72} \int \frac{1}{\sqrt{u}} du \right] \\ &= x \arcsin(6x) + \frac{1}{12} [2\sqrt{u}] + C \\ &= x \arcsin(6x) + \frac{1}{6} \sqrt{1 - 36x^2} + C.\end{aligned}$$

2. (a) We use u -substitution. Let $u = x^2 - 9$ so $du = 2x dx$ and $\frac{1}{2} du = x dx$. The integral becomes

$$\begin{aligned}\int \frac{x}{\sqrt{x^2 - 9}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \\ &= \frac{1}{2} \int u^{-\frac{1}{2}} du \\ &= \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= \sqrt{x^2 - 9} + C.\end{aligned}$$

(b) This is an arcsecant type integral with $k = 3$, so

$$\int \frac{1}{x\sqrt{x^2 - 9}} dx = \int \frac{1}{x\sqrt{x^2 - 3^2}} dx = \frac{1}{3} \operatorname{arcsec}\left(\frac{x}{3}\right) + C.$$

(c) We use integration by parts. Let $w = x$ so $dw = dx$, and let $dv = \csc^2(9x) dx$ so $v = -\frac{1}{9} \cot(9x)$. Then

$$\begin{aligned}\int x \csc^2(9x) dx &= -\frac{1}{9} x \cot(9x) + \frac{1}{9} \int \cot(9x) dx \\ &= -\frac{1}{9} x \cot(9x) + \frac{1}{9} \cdot \frac{1}{9} \ln|\sin(9x)| + C \\ &= -\frac{1}{9} x \cot(9x) + \frac{1}{81} \ln|\sin(9x)| + C.\end{aligned}$$

(d) We use u -substitution. Let $u = x^5$ so $du = 5x^4 dx$ and $\frac{1}{5} du = x^4 dx$. The integral becomes

$$\begin{aligned}\int x^4 e^{x^5} dx &= \frac{1}{5} \int e^u du \\ &= \frac{1}{5} e^u + C \\ &= \frac{1}{5} e^{x^5} + C.\end{aligned}$$

- (e) We begin with u -substitution. Let $u = x^5$ so $du = 5x^4 dx$ and $\frac{1}{5} du = x^4 dx$. The integral becomes

$$\int x^9 e^{x^5} dx = \int x^5 e^{x^5} (x^4 dx) = \frac{1}{5} \int u e^u du.$$

Now we use integration by parts, letting $w = u$ so $dw = du$, and $dv = e^u du$ so $v = e^u$. We obtain

$$\begin{aligned} \int x^9 e^{x^5} dx &= \frac{1}{5} \left[u e^u - \int e^u du \right] \\ &= \frac{1}{5} u e^u - \frac{1}{5} e^u + C \\ &= \frac{1}{5} x^5 e^{x^5} - \frac{1}{5} e^{x^5} + C. \end{aligned}$$

- (f) We begin by completing the square:

$$\begin{aligned} 9x^2 - 12x + 8 &= 9 \left[\left(x^2 - \frac{4}{3}x \right) + \frac{8}{9} \right] \\ &= 9 \left[\left(x^2 - \frac{4}{3}x + \frac{4}{9} \right) + \frac{8}{9} - \frac{4}{9} \right] \\ &= 9 \left[\left(x - \frac{2}{3} \right)^2 + \frac{4}{9} \right] \\ &= (3x - 2)^2 + 4. \end{aligned}$$

Thus the integral becomes

$$\int \frac{1}{9x^2 - 12x + 8} dx = \frac{1}{(3x - 2)^2 + 4} dx.$$

Now let $u = 3x - 2$ so $du = 3 dx$ and $\frac{1}{3} du = dx$. Finally,

$$\begin{aligned} \int \frac{1}{9x^2 - 12x + 8} dx &= \frac{1}{3} \int \frac{1}{u^2 + 4} du \\ &= \frac{1}{3} \cdot \frac{1}{2} \arctan \left(\frac{u}{2} \right) + C \\ &= \frac{1}{6} \arctan \left(\frac{3x - 2}{2} \right) + C. \end{aligned}$$

- (g) We try integration by parts, with $w = e^{4x}$ so $dw = 4e^{4x} dx$ and $dv = \cos(x) dx$ so $v = \sin(x)$. Then

$$\int e^{4x} \cos(x) dx = e^{4x} \sin(x) - 4 \int e^{4x} \sin(x) dx.$$

We try integration by parts a second time. Again, we let $w = e^{4x}$ so $dw = 4e^{4x} dx$, and now we let $dv = \sin(x) dx$ so $v = -\cos(x)$. Thus

$$\begin{aligned} \int e^{4x} \cos(x) dx &= e^{4x} \sin(x) - 4 \left[-e^{4x} \cos(x) + 4 \int e^{4x} \cos(x) dx \right] \\ &= e^{4x} \sin(x) + 4e^{4x} \cos(x) - 16 \int e^{4x} \cos(x) dx \\ 17 \int e^{4x} \cos(x) dx &= e^{4x} \sin(x) + 4e^{4x} \cos(x) + C \\ \int e^{4x} \cos(x) dx &= \frac{1}{17} e^{4x} \sin(x) + \frac{4}{17} e^{4x} \cos(x) + C. \end{aligned}$$

(h) We begin by performing long division:

$$\begin{array}{r} 6x - 1 \\ 2x - 5 \overline{) 12x^2 - 32x + 14} \\ \underline{12x^2 - 30x} \\ -2x + 14 \\ \underline{-2x + 5} \\ 9 \end{array}$$

Now we can write

$$\begin{aligned} \int \frac{12x^2 - 32x + 14}{2x - 5} dx &= \int \left(6x - 1 + \frac{9}{2x - 5} \right) dx \\ &= 3x^2 - x + \frac{9}{2} \ln|2x - 5| + C. \end{aligned}$$

(i) Let $u = \ln(x)$ so $du = \frac{1}{x} dx$. The integral becomes

$$\begin{aligned} \int \frac{1}{x\sqrt{4 - \ln^2(x)}} dx &= \int \frac{1}{\sqrt{4 - u^2}} du \\ &= \arcsin\left(\frac{u}{2}\right) + C \\ &= \arcsin\left(\frac{\ln(x)}{2}\right) + C. \end{aligned}$$

(j) Recall that $1 + \tan^2(x) = \sec^2(x)$, so

$$\int \cos^2(x)[1 + \tan^2(x)] dx = \int \cos^2(x) \sec^2(x) dx = \int dx = x + C.$$

Alternatively, we could write

$$\int \cos^2(x)[1 + \tan^2(x)] dx = \int [\cos^2(x) + \sin^2(x)] dx = \int dx = x + C.$$

3. (a) Let $w = \sin^{n-1}(x)$ so $dw = (n-1)\sin^{n-2}(x)\cos(x)dx$. Let $dv = \sin(x)dx$ so $v = -\cos(x)$. Then

$$\begin{aligned}\int \sin^n(x) dx &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x) dx \\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)[1 - \sin^2(x)] dx \\ &= -\cos(x)\sin^{n-1}(x) \\ &\quad + (n-1)\int \sin^{n-2}(x) dx - (n-1)\int \sin^n(x) dx \\ n\int \sin^n(x) dx &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x) dx \\ \int \sin^n(x) dx &= -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x) dx.\end{aligned}$$

- (b) Using the reduction formula with $n = 7$, we obtain

$$\int \sin^7(x) dx = -\frac{1}{7}\cos(x)\sin^6(x) + \frac{6}{7}\int \sin^5(x) dx.$$

Now using the formula with $n = 5$, this becomes

$$\begin{aligned}\int \sin^7(x) dx &= -\frac{1}{7}\cos(x)\sin^6(x) + \frac{6}{7}\left[-\frac{1}{5}\cos(x)\sin^4(x) + \frac{4}{5}\int \sin^3(x) dx\right] \\ &= -\frac{1}{7}\cos(x)\sin^6(x) - \frac{6}{35}\cos(x)\sin^4(x) + \frac{24}{35}\int \sin^3(x) dx.\end{aligned}$$

We use the formula once more, this time with $n = 3$, and get

$$\begin{aligned}\int \sin^7(x) dx &= -\frac{1}{7}\cos(x)\sin^6(x) - \frac{6}{35}\cos(x)\sin^4(x) \\ &\quad + \frac{24}{35}\left[-\frac{1}{3}\cos(x)\sin^2(x) + \frac{2}{3}\int \sin(x) dx\right] \\ &= -\frac{1}{7}\cos(x)\sin^6(x) - \frac{6}{35}\cos(x)\sin^4(x) - \frac{8}{35}\cos(x)\sin^2(x) - \frac{16}{35}\cos(x) + C.\end{aligned}$$