Section 4.2: Separable Equations

A DE is <u>separable</u> if it can be written such that all expressions involving the unknown function y appear only on one side of the equation, and all expressions involving the independent variable t appear only on the other side. To do this we use the relationship between differentials given by

$$dy = \frac{dy}{dt}\,dt \quad \text{or} \quad dy = y'\,dt.$$

Then we integrate both sides to solve the DE.

eg $$t^4 \frac{dy}{dt} - \frac{1}{y^2} = 0$$

We can rewrite this DE as

$$t^4 \frac{dy}{dt} = \frac{1}{y^2}$$

$$y^2 \frac{dy}{dt} = t^{-4}$$

$$y^2\,dy = t^{-4}\,dt$$

Since we have successfully separated the variables y and t, this is a separable DE.

Now we integrate both sides:

$$\int y^2\,dy = \int t^{-4}\,dt$$

$$\frac{y^3}{3} = \frac{t^{-3}}{-3} + C$$

$$y^3 = -\frac{1}{t^3} + C$$

$$\boxed{y = \sqrt[3]{C - \frac{1}{t^3}}}$$

We are often unable to express the solution of a separable DE as an explicit function. Instead, it is common to write these solutions in an implicit form.

eg Solve the IVP

$$ty(y^2+1)\frac{dy}{dt} = 7, \quad y(1) = 0$$

This is a separable DE:

$$y(y^2+1)\,dy = \frac{7}{t}\,dt$$

$$\int (y^3+y)\,dy = 7\int \frac{1}{t}\,dt$$

$$\frac{y^4}{4} + \frac{y^2}{2} = 7\ln|t| + C \quad \text{(general solution)}$$

$$y^4 + 2y^2 = 28\ln|t| + C$$

$$y^4 + 2y^2 - 28\ln|t| = C$$

Since $y(1) = 0$, this becomes $0 + 0 - 28\ln(1) = C$

so $C = 0$. Hence the particular solution is

$$\boxed{y^4 + 2y^2 - 28\ln|t| = 0}.$$

eg Recall the DE $\frac{dy}{dt} = ky$. Observe that this is a separable DE because it can be written

$$\frac{1}{y}\,dy = k\,dt$$

$$\int \frac{1}{y}\,dy = k\int dt$$

$$\ln|y| = kt + C$$

We can assume that $y > 0$, so the general solution becomes

$$\ln(y) = kt + C$$

$$e^{\ln(y)} = e^{kt+C}$$

$$y = e^{kt} \cdot e^{C} = Ce^{kt}$$

Further assume that we have the initial condition given by $y(0) = y_0$. Now we know that

$$y(0) = Ce^{0} = C = y_0$$

so the particular solution is given by

$$y = y_0 e^{kt}.$$

Often we are interested in using the solutions of a DE to help us understand the overall behaviour of the model, called its <u>dynamics</u>. In particular, we often want to know what happens as $t \to \infty$.

To explore the dynamics of the DE $\frac{dy}{dt} = ky$, we can consider the limit as $t \to \infty$ of its solution.

If $k > 0$ we have

$$\lim_{t \to \infty} y = \lim_{t \to \infty} y_0 e^{kt} = \infty$$

which is exponential growth.

If $k < 0$ then

$$\lim_{t \to \infty} y = \lim_{t \to \infty} y_0 e^{kt} = 0$$

which is exponential decay.

If $k = 0$ then the solution of the DE is

$$y = y_0 e^{0 \cdot t} = y_0$$

$$\lim_{t \to \infty} y = \lim_{t \to \infty} y_0 = y_0$$

We can also determine these dynamics directly from the DE.

If $k > 0$, then $\frac{dy}{dt} = ky > 0$ so y is always increasing.

If $k < 0$, then $\frac{dy}{dt} = ky < 0$ so y is always decreasing.

If $k = 0$, then $\frac{dy}{dt} = ky = 0$ so y does not change.

This is qualitative analysis of the DE.

eg A colony of 100 rabbits is brought to a large island with abundant food and no predators, such that its rate of change is proportional to its current size. After 2 years, there are 250 rabbits. Determine the size of the population after 6 years.

Here $y_0 = y(0) = 100$ and $y(2) = 250$.

Thus $y = 100e^{kt}$ and so

$$y(2) = 100e^{k \cdot 2} = 250$$

$$e^{2k} = \frac{250}{100} = \frac{5}{2}$$

$$\ln(e^{2k}) = \ln(5/2)$$

$$2k = \ln(5/2)$$

$$k = \frac{1}{2}\ln(5/2)$$

Now we have $y = 100e^{\frac{1}{2}\ln(5/2)t}$

$$y(6) = 100e^{\frac{1}{2}\ln(5/2) \cdot 6}$$

$$= 100e^{3\ln(5/2)}$$

$$= 100e^{\ln((5/2)^3)}$$

$$= 100e^{\ln(125/8)}$$

$$= 100 \cdot \frac{125}{8} \approx 1563$$

After 6 years, there are about <u>1563 rabbits</u>.

eg A patient has a bacterial infection. A course of antibiotics causes the bacteria to undergo exponential decay, such that after 4 days two-fifths of the bacteria have been eliminated. Find the fraction of the original bacteria population that remains after 8 days.

If y_0 is the initial population then we are given that $y(4) = y_0 - \frac{2}{5}y_0 = \frac{3}{5}y_0$. This means that

$$y = y_0 e^{kt}$$

$$y(4) = \frac{y_0 e^{k\cdot 4}}{y_0} = \frac{\frac{3}{5}y_0}{y_0}$$

$$e^{4k} = \frac{3}{5}$$

$$\ln(e^{4k}) = \ln(3/5)$$

$$4k = \ln(3/5)$$

$$k = \frac{1}{4}\ln(3/5)$$

Then $y = y_0 e^{\frac{1}{4}\ln(3/5)t}$

$$y(8) = y_0 e^{\frac{1}{4}\ln(3/5)\cdot 8}$$

$$= y_0 e^{2\ln(3/5)}$$

$$= y_0 e^{\ln(9/25)}$$

$$= y_0 \cdot \frac{9}{25}$$

Hence $\frac{9}{25}$ of the original bacteria remains after 8 days.

eg Radioactive elements decay spontaneously at a rate proportional to the mass of the sample. Hence they undergo exponential decay. The <u>half-life</u> measures the time it takes for half the sample to decay. Carbon-14 has a half-life of 5730 years. Determine, to the nearest year, how long it takes for a sample of 100 mg of Carbon-14 to decay to 10 mg.

We have $y_0 = 100$ and we know that $y(5730) = 50$.

Then $y = y_0 e^{kt}$

$$= 100e^{kt}$$

$$y(5730) = 100e^{k \cdot 5730} = 50$$

$$e^{5730k} = \frac{1}{2}$$

$$\ln(e^{5730k}) = \ln(1/2) = \ln(2^{-1})$$

$$5730k = -\ln(2)$$

$$k = \frac{-\ln(2)}{5730}$$

Now we have $y = 100e^{-\frac{\ln(2)}{5730}t}$ so we set

$$10 = 100e^{-\frac{\ln(2)}{5730}t}$$

$$\frac{1}{10} = e^{-\frac{\ln(2)}{5730}t}$$

$$-\ln(10) = -\frac{\ln(2)}{5730}t$$

$$t = \frac{5730\ln(10)}{\ln(2)} \approx 19036$$

It takes about 19,036 years for the sample to decay to 10 mg.