Section 3.2: Trigonometric Integrals

Target: Integrals of powers of trigonometric functions and their products.

We will focus on integrals of the form $\int \sin^m(x)\cos^n(x)dx$.

① Suppose m is odd. We factor aside one $\sin(x)$ to serve as part of the differential in u-substitution. We then transform any remaining $\sin(x)$ into $\cos(x)$ using the identity $\sin^2(x)+\cos^2(x)=1$, so $\sin^2(x)=1-\cos^2(x)$. Now we let $u=\cos(x)$ so $du=-\sin(x)dx$ and $-du=\sin(x)dx$.

eg $\int \sin^3(x)\cos^6(x)dx$

$$= \int \sin^2(x)\cos^6(x)\cdot\sin(x)dx$$

$$= \int \left[1-\cos^2(x)\right]\cos^6(x)\cdot\sin(x)dx$$

Let $u=\cos(x)$
$-du=\sin(x)dx$

$$= -\int \left[1-u^2\right]u^6\,du$$

$$= -\int (u^6-u^8)\,du$$

$$= -\left[\frac{u^7}{7}-\frac{u^9}{9}\right]+C$$

$$\boxed{= \frac{1}{9}\cos^9(x)-\frac{1}{7}\cos^7(x)+C}$$

(2) Suppose n is odd. We factor aside one $\cos(x)$ to serve as part of the differential in u-substitution. We then transform any remaining $\cos(x)$ into $\sin(x)$ using the identity $\cos^2(x) = 1 - \sin^2(x)$. Now we let $u = \sin(x)$ so $du = \cos(x)\,dx$.

eg $\int \sin^6(3x+1)\cos^5(3x+1)\,dx$

$$= \int \sin^6(3x+1)\cos^4(3x+1) \cdot \cos(3x+1)\,dx$$

$$= \int \sin^6(3x+1)\left[\cos^2(3x+1)\right]^2 \cdot \cos(3x+1)\,dx$$

$$= \int \sin^6(3x+1)\left[1 - \sin^2(3x+1)\right]^2 \cdot \cos(3x+1)\,dx$$

We let $u = \sin(3x+1)$ so $du = 3\cos(3x+1)\,dx$

$$\frac{1}{3}du = \cos(3x+1)\,dx$$

The integral becomes

$$\frac{1}{3}\int u^6\left[1-u^2\right]^2 du = \frac{1}{3}\int \left(u^6 - 2u^8 + u^{10}\right) du$$

$$= \frac{1}{3}\left[\frac{u^7}{7} - 2\cdot\frac{u^9}{9} + \frac{u^{11}}{11}\right] + C$$

$$\boxed{= \frac{1}{21}\sin^7(3x+1) - \frac{2}{27}\sin^9(3x+1) + \frac{1}{33}\sin^{11}(3x+1) + C}$$

These strategies will still work even if only one of $\sin(x)$ or $\cos(x)$ is present, as long as its power is odd.

eg
$$\int_0^{\pi/12} \cos^3(2x)\,dx = \int_0^{\pi/12} \cos^2(2x)\cdot\cos(2x)\,dx$$
$$= \int_0^{\pi/12} \left[1-\sin^2(2x)\right]\cdot\cos(2x)\,dx$$

Let $u = \sin(2x)$
$du = 2\cos(2x)\,dx$
$\frac{1}{2}du = \cos(2x)\,dx$
When $x=0$, $u=\sin(0)=0$
$x=\frac{\pi}{12}$, $u=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

$$= \frac{1}{2}\int_0^{1/2} \left[1-u^2\right]du$$
$$= \frac{1}{2}\left[u-\frac{u^3}{3}\right]_0^{1/2}$$
$$= \frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{24}\right)-0\right]$$
$$\boxed{= \frac{11}{48}}$$

(3) If both m and n are odd then we can follow the strategy of (1) or (2). We normally choose the strategy in which u will be whichever of $\sin(x)$ or $\cos(x)$ that has the larger power.

(4) Now suppose that both m and n are even.
We apply the half-angle identities

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2(x) = \frac{1-\cos(2x)}{2}.$$

eg $\int \cos^2(x-3)\,dx = \int \frac{1+\cos(2\cdot(x-3))}{2}\,dx$

$$= \frac{1}{2}\int \left[1+\cos(2x-6)\right]dx$$

$$= \frac{1}{2}\left[x + \frac{\sin(2x-6)}{2}\right] + C$$

$$\boxed{= \frac{1}{2}x + \frac{1}{4}\sin(2x-6) + C}$$

eg $\int \sin^4(x)\,dx = \int \left[\sin^2(x)\right]^2 dx$

$$= \int \left[\frac{1-\cos(2x)}{2}\right]^2 dx$$

$$= \int \frac{1-2\cos(2x)+\cos^2(2x)}{4}\,dx$$

$$= \frac{1}{4}\int \left[1-2\cos(2x) + \frac{1+\cos(4x)}{2}\right]dx$$

$$= \frac{1}{4}\int \left[\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right]dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - 2\cdot\frac{\sin(2x)}{2} + \frac{1}{2}\cdot\frac{\sin(4x)}{4}\right] + C$$

$$\boxed{= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$$