Section 2.2: Definite Integration

Def'n: If $f(x)$ is defined on a closed interval $[a,b]$ and $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ exists then we say that $f(x)$ is integrable on $[a,b]$.

Furthermore, we write

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

and we call this the definite integral of $f(x)$ on $[a,b]$. Here, $x=a$ is the lower bound of the definite integral and $x=b$ is its upper bound.

eg Given $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 - e^{x_i})\Delta x$ on $[0,4]$ we can rewrite the limit of the Riemann sum as the definite integral

$$\int_0^4 (x^2 - e^x)\,dx.$$

eg Evaluate the definite integral $\int_{-3/2}^{3} (2x+3)\,dx$.

We can write

$$\int_{-3/2}^{3} (2x+3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

where $\Delta x_i = \Delta x = \dfrac{3-(-3/2)}{n} = \dfrac{9/2}{n} = \dfrac{9}{2n}$

$$x_i^* = x_i = -\frac{3}{2} + \frac{9i}{2n}$$

$$f(x_i^*) = 2\left(-\frac{3}{2}+\frac{9i}{2n}\right) + 3 = -3 + \frac{9i}{n} + 3 = \frac{9i}{n}$$

So $\displaystyle\int_{-3/2}^{3} (2x+3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9i}{n}\cdot\frac{9}{2n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{81i}{2n^2}$$

$$= \lim_{n\to\infty} \left[\frac{81}{2n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{2n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{81(n+1)}{4n}$$

$$\boxed{= \frac{81}{4}}$$

Consider a function $f(x)$ which is continuous and non-negative on $[a, b]$, with $a < b$. Then the area A of the region which lies below the curve $y = f(x)$, above the x-axis, and between $x = a$ and $x = b$ can be written

$$A = \int_a^b f(x)\,dx .$$

eg Because $f(x) = 2x + 3$ is continuous and non-negative on $[-\frac{3}{2}, 3]$, the last example shows that

$$A = \int_{-3/2}^{3} (2x+3)\,dx \boxed{= \frac{81}{4}} .$$

However, in terms of a definite integral, we can consider $a \geq b$.

If $a = b$ then we have $\int_a^a f(x)\,dx$ where

$$\Delta x = \frac{b-a}{n} = \frac{a-a}{n} = \frac{0}{n} = 0$$

So
$$\begin{aligned} \int_a^a f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} 0 \\ &= \lim_{n\to\infty} 0 \\ &= 0, \text{ if } f(a) \text{ is defined.} \end{aligned}$$

If $a > b$, then observe that

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $\Delta x = \dfrac{a-b}{n}$ so that

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{a-b}{n}$$

$$= -\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n}$$

$$= -\int_a^b f(x)\,dx$$

eg We have shown that $\int_{-3/2}^{3} (2x+3)\,dx = \dfrac{81}{4}$ so

$$\int_3^{-3/2} (2x+3)\,dx \boxed{= -\frac{81}{4}}.$$

Theorem : Basic Properties of Definite Integrals

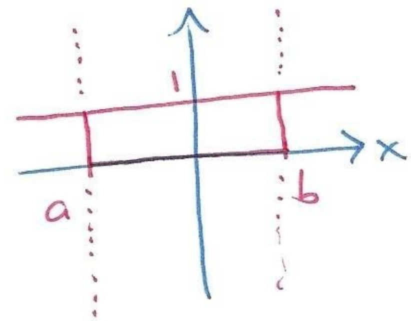
1. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

We have a common definite integral:

$$\int_a^b 1 \cdot dx = \int_a^b dx$$

$$= (b-a) \cdot 1$$

$$= b-a$$



eg $\displaystyle\int_{-\pi/2}^{5\sqrt{3}} dx = 5\sqrt{3} - (-\pi/2) \boxed{= 5\sqrt{3} + \frac{\pi}{2}}$

eg Given that $\displaystyle\int_0^{\pi/2} \cos(x)\,dx = 1$, evaluate

$$\int_0^{\pi/2} [4 - 7\cos(x)]\,dx.$$

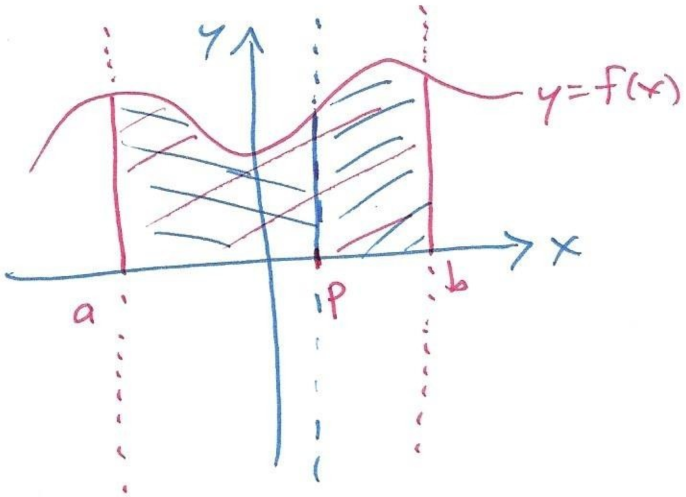
Using the Basic Properties, we can rewrite the definite integral:

$$\int_0^{\pi/2} [4 - 7\cos(x)]\,dx = \int_0^{\pi/2} 4\,dx - \int_0^{\pi/2} 7\cos(x)\,dx$$

$$= 4\int_0^{\pi/2} dx - 7\int_0^{\pi/2} \cos(x)\,dx$$

$$= 4\left[\frac{\pi}{2} - 0\right] - 7 \cdot 1$$

$$\boxed{= 2\pi - 7}$$



Theorem: The Additive Interval Property

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$

eg Given that $\int_0^2 x^2\,dx = \frac{8}{3}$ and $\int_2^3 x\,dx = \frac{5}{2}$,

evaluate $\int_0^3 f(x)\,dx$ where

$$f(x) = \begin{cases} 3x^2 & \text{for } x < 2 \\ 6x & \text{for } x \geq 2. \end{cases}$$

Using the Additive Interval Property, we can write

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_0^2 3x^2\,dx + \int_2^3 6x\,dx$$

$$= 3\int_0^2 x^2\,dx + 6\int_2^3 x\,dx$$

$$= 3\cdot\frac{8}{3} + 6\cdot\frac{5}{2}$$

$$\boxed{=23}$$