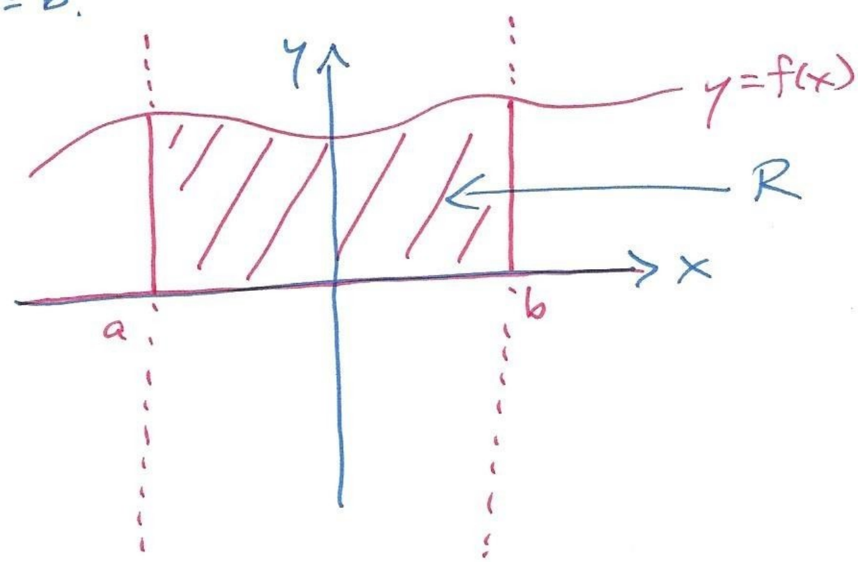
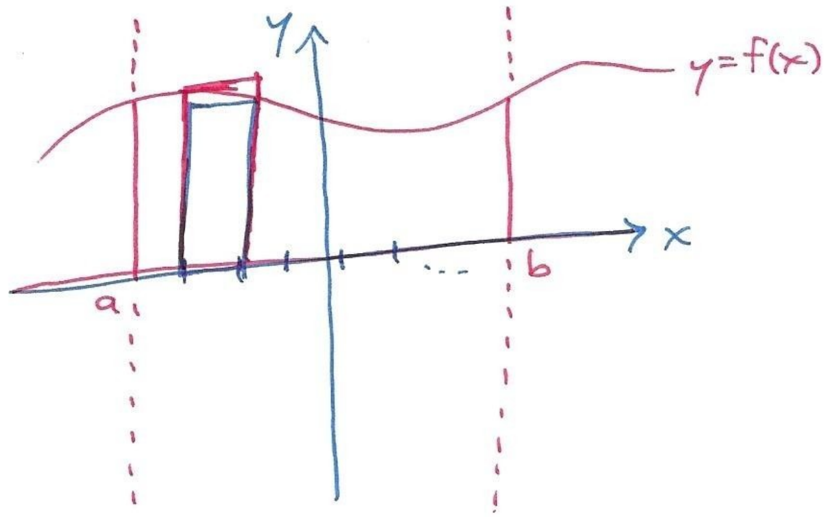
Section 2.1: Area Under a Curve

We know how to find the area of elementary shapes like rectangles, triangles and circles. But how can we find the area of a region that is <u>not</u> simply a combination of these shapes?

Specifically, consider a region R which is bounded above by a curve $y = f(x)$, bounded below by the x-axis, bounded to the left by $x = a$, and bounded to the right by $x = b$.



We want to determine the exact value of the area A of the region R.

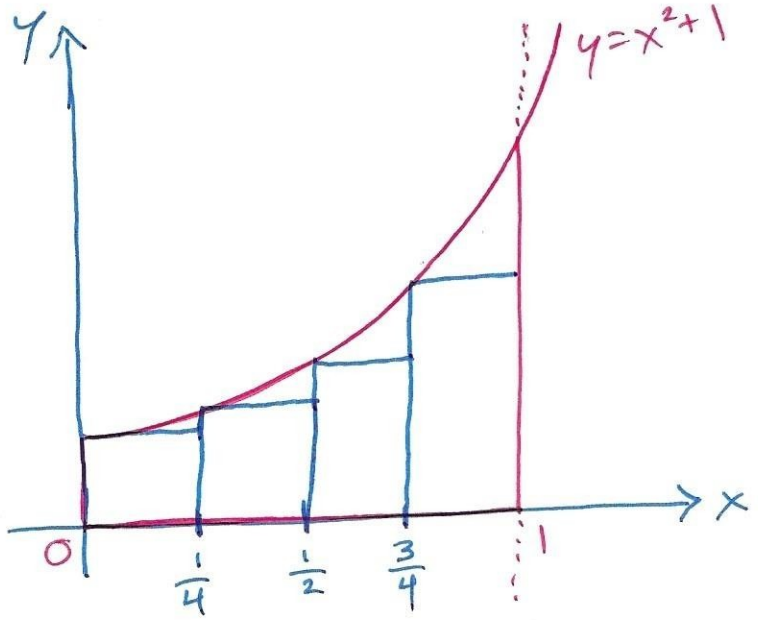


In order to find an approximation of the area A, we will approximate the region R with a number n of rectangles. Then we could compute the area of each rectangle, and estimate A by taking their sum.

We will divide the interval $[a,b]$ or $a \le x \le b$ into n <u>subintervals</u>, and use those subintervals as one side of each rectangle. We call this a <u>partition</u> of $[a,b]$.

If each subinterval has the same width Δx then we call it a <u>regular partition</u>.

eg Estimate the area bounded above by $f(x) = x^2+1$, below by the x-axis, to the left by $x=0$, and to the right by $x=1$. Do so with a regular partition into 4 subintervals.



We want to divide $[0,1]$ into 4 equal subintervals. Thus they will have width

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

which will also be the width of each rectangle.

Thus the first rectangle will be drawn on the subinterval $[0, 1/4]$.

" second rectangle " $[1/4, 1/2]$.

" third rectangle " $[1/2, 3/4]$.

" fourth rectangle " $[3/4, 1]$.

For the height of each rectangle, we will choose the minimum value of $f(x)$ on the corresponding subinterval. These are called <u>inscribed</u> rectangles.

For the first rectangle, its height will be $f(0) = 1$, so its area is $A_1 = \frac{1}{4} \cdot 1 = \frac{1}{4}$.

For the second rectangle, its height will be $f(\frac{1}{4}) = \frac{17}{16}$, so its area is $A_2 = \frac{1}{4} \cdot \frac{17}{16} = \frac{17}{64}$.

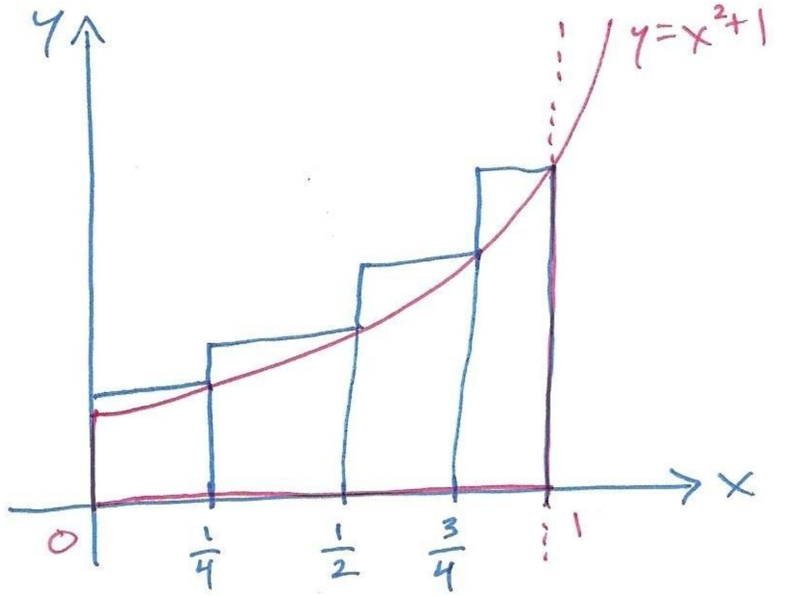
For the third rectangle, its height will be $f(\frac{1}{2}) = \frac{5}{4}$, so its area is $A_3 = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}$.

For the fourth rectangle, its height will $f(\frac{3}{4}) = \frac{25}{16}$, so its area is $A_4 = \frac{1}{4} \cdot \frac{25}{16} = \frac{25}{64}$.

Then we can estimate the true area A to be

$$A \approx A_1 + A_2 + A_3 + A_4 = \frac{39}{32} \approx 1.22.$$

This is the smallest reasonable estimate of A, and is therefore called the lower sum.



Instead, we could try using the maximum value of $f(x)$ on the corresponding interval to give the height of each rectangle. These are called circumscribed rectangles.

For the first rectangle, its height will $f(\frac{1}{4}) = \frac{17}{16}$, so its area is $B_1 = \frac{1}{4} \cdot \frac{17}{16} = \frac{17}{64}$.

Likewise, the other rectangles will have heights of $f(\frac{1}{2}) = \frac{5}{4}$, $f(\frac{3}{4}) = \frac{25}{16}$ and $f(1) = 2$ so their areas will be

$$B_2 = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}, \quad B_3 = \frac{1}{4} \cdot \frac{25}{16} = \frac{25}{64}, \quad B_4 = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

Our new estimate ~~of~~ of A is

$$A \approx B_1 + B_2 + B_3 + B_4 = \frac{47}{32} \approx 1.47.$$

This is the largest reasonable estimate of A, and is called the <u>upper sum</u>.

We can conclude that $\frac{39}{32} \leq A \leq \frac{47}{32}$.

Suppose we want to approximate A using n rectangles.

Then we partition the interval $[a, b]$ into n subintervals of width

$$\Delta x = \frac{b-a}{n}.$$

Then the first subinterval is $[a, a+\Delta x]$

second subinterval is $[a+\Delta x, a+2\Delta x]$

$\vdots$

ith subinterval is $[a+(i-1)\Delta x, a+i\Delta x]$.

$\vdots$

nth subinterval is $[a+(n-1)\Delta x, a+n\Delta x]$

where $a+n\Delta x = a+n\cdot\frac{b-a}{n} = a+(b-a) = b.$

We call the right endpoint of the ith subinterval x_i

where

$$x_i = a+i\Delta x$$

Note that the left endpoint of the ith subinterval is, there fore, x_{i-1}.

We can then denote $x_0 = a$ and $x_n = b$.

Now we assume that $f(x)$ is continuous on $[a, b]$ and therefore on each subinterval. Then the Extreme Value Theorem guarantees that $f(x)$ will have both a minimum value and a maximum value on each subinterval.

Let $x = m_i$ be the absolute minimum of $f(x)$ on the ith subinterval, so $f(m_i)$ is the minimum value. Then we would choose the height of each rectangle to be $f(m_i)$ in order to compute the lower sum.

Similarly let $x = M_i$ be the absolute maximum of $f(x)$ on the ith subinterval, so $f(M_i)$ is the maximum value. Then we choose $f(M_i)$ to be the height of the rectangles to obtain the upper sum.

In general, then, the lower sum is given by

$$s(n) = f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + \cdots + f(m_n)\Delta x$$
$$= [f(m_1) + f(m_2) + f(m_3) + \cdots + f(m_n)]\Delta x$$

The upper sum is given by

$$S(n) = f(M_1)\Delta x + f(M_2)\Delta x + f(M_3)\Delta x + \cdots + f(M_n)\Delta x$$
$$= [f(M_1) + f(M_2) + f(M_3) + \cdots + f(M_n)]\Delta x$$

Consider a sum of n terms $a_1, a_2, a_3, \ldots, a_n$ for which all of the terms can be described by a general formula a_i. Here we call i the index of summation. Then we can write the sum in sigma notation as follows:

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Here, we call a_i the summand, $i=1$ the lower bound of summation, and n the upper bound of summation.

eg $1 + 4 + 9 + 16 + \cdots + 144$

All of the terms can be written in the form i^2. Thus $a_i = i^2$ with lower bound 1 and upper bound 12. Hence we can write this sum in sigma notation as $\sum_{i=1}^{12} i^2$.

eg $2 + \frac{3}{8} + \frac{4}{27} + \frac{5}{64} + \cdots + \frac{n+1}{n^3} = \sum_{i=1}^{n} \frac{i+1}{i^3}$

Theorem: Basic Properties of Sums

1. $\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$ for any constant k

2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

3. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

However, $\sum_{i=1}^{n} a_i b_i \neq \left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} b_i\right)$

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \neq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

eg
$$\sum_{i=1}^{n} (2i - 1) = \sum_{i=1}^{n} 2i - \sum_{i=1}^{n} 1$$
$$= 2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

Theorem: Summation Formulas

1. $\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n$

2. $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

eg Find an expression for $\sum_{i=1}^{n}(2i-1)$ and use it to compute $\sum_{i=1}^{100}(2i-1)$.

We have already seen that

$$\sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$= 2\cdot\frac{n(n+1)}{2} - n$$

$$= n^2 + n - n$$

$$\boxed{= n^2}$$

Next, $\sum_{i=1}^{100}(2i-1) = 100^2 \boxed{= 10000}$.

eg Evaluate $\sum_{i=1}^{n}(ni-4)^2$

We can write

$$\sum_{i=1}^{n}(ni-4)^2 = \sum_{i=1}^{n}(n^2i^2 - 8ni + 16)$$

$$= n^2\sum_{i=1}^{n} i^2 - 8n\sum_{i=1}^{n} i + 16\sum_{i=1}^{n} 1$$

$$= n^2\cdot\frac{n(n+1)(2n+1)}{6} - 8n\cdot\frac{n(n+1)}{2} + 16\cdot n$$

$$\boxed{= \frac{1}{3}n^5 + \frac{1}{2}n^4 - \frac{23}{6}n^3 - 4n^2 + 16n}$$

Using sigma notation, we can rewrite the lower sum as

$$s(n) = [f(m_1) + f(m_2) + f(m_3) + \cdots + f(m_n)]\Delta x$$
$$= \left[\sum_{i=1}^{n} f(m_i)\right]\Delta x$$
$$= \sum_{i=1}^{n} f(m_i)\Delta x .$$

Likewise, the upper sum becomes

$$S(n) = [f(M_1) + f(M_2) + f(M_3) + \cdots + f(M_n)]\Delta x$$
$$= \left[\sum_{i=1}^{n} f(M_i)\right]\Delta x$$
$$= \sum_{i=1}^{n} f(M_i)\Delta x$$

eg Let's again consider the region R under $f(x) = x^2+1$, above the x-axis, between $x=0$ and $x=1$.

We will find expressions for the lower sum and the upper sum using n subintervals, and use them to estimate the area A using 100 subintervals.

Assuming a regular partition, $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$.

Then the right endpoint of each subinterval is

$$x_i = 0 + i\Delta x = \frac{i}{n}$$

and thus the left endpoint is $x_{i-1} = \dfrac{i-1}{n}$.

Since $f(x)$ is increasing the minimum value on each subinterval must occur at the left endpoint x_{i-1} so

$$m_i = \frac{i-1}{n}$$

$$f(m_i) = \left(\frac{i-1}{n}\right)^2 + 1 = \frac{i^2}{n^2} - \frac{2i}{n^2} + \frac{1}{n^2} + 1$$

Thus the lower sum is given by

$$s(n) = \sum_{i=1}^{n} \left(\frac{i^2}{n^2} - \frac{2i}{n^2} + \frac{1}{n^2} + 1\right) \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{i^2}{n^3} - \frac{2i}{n^3} + \frac{1}{n^3} + \frac{1}{n}\right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 - \frac{2}{n^3} \sum_{i=1}^{n} i + \frac{1}{n^3} \sum_{i=1}^{n} 1 + \frac{1}{n} \sum_{i=1}^{n} 1$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{2}{n^3} \cdot \frac{n(n+1)}{2} + \frac{1}{n^3} \cdot n + \frac{1}{n} \cdot n$$

$$= \frac{4}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

$s(100) \approx 1.328$ Compare: $s(4) = 1.22$

The maximum value on each subinterval occurs at the right endpoint x_i so

$$M_i = \frac{i}{n}$$

$$f(M_i) = \left(\frac{i}{n}\right)^2 + 1 = \frac{i^2}{n^2} + 1$$

The upper sum will be

$$S'(n) = \sum_{i=1}^{n} \left(\frac{i^2}{n^2} + 1\right) \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{i^2}{n^3} + \frac{1}{n}\right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 + \frac{1}{n} \sum_{i=1}^{n} 1$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{n} \cdot n$$

$$= \frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Compare: $S'(4) = 1.47$

$S'(100) \approx 1.338$

In fact, observe that

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \left(\frac{4}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{4}{3} - 0 + 0$$

$$= \frac{4}{3}$$

$$\lim_{n\to\infty} S'(n) = \lim_{n\to\infty} \left(\frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{4}{3} + 0 + 0$$

$$= \frac{4}{3}$$

Thus we can conclude that

$$A = \lim_{n\to\infty} s(n) = \lim_{n\to\infty} S'(n) \boxed{= \frac{4}{3}}$$

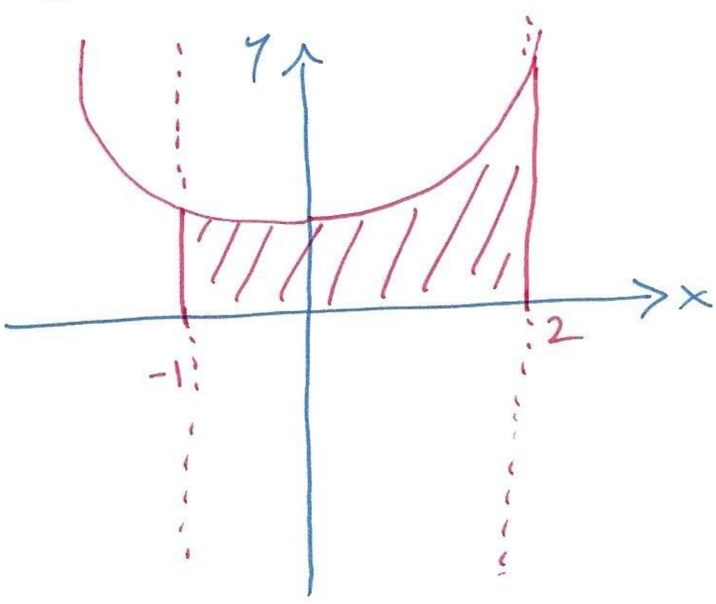
To compute A, we can choose any <u>sample point</u> x_i^* which lies on the appropriate subinterval and set the height of each rectangle to be $f(x_i^*)$. Since we have freedom in terms of how we choose x_i^*, we typically want it to have the simplest possible form, so we normally choose $x_i^* = x_i$ (the right endpoint).

In general, then we have found that

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* = a + i\Delta x$.

eg Find the area under $f(x) = x^2+1$, above the x-axis, and between $x=-1$ and $x=2$.



We will choose

$$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i^* = x_i = -1 + \frac{3i}{n}$$

$$f(x_i^*) = \left(-1+\frac{3i}{n}\right)^2 + 1$$

$$= 1 - \frac{6i}{n} + \frac{9i^2}{n^2} + 1$$

$$= \frac{9i^2}{n^2} - \frac{6i}{n} + 2$$

Then $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{6i}{n} + 2 \right) \cdot \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^2}{n^3} - \frac{18i}{n^2} + \frac{6}{n} \right)$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{18}{n^2} \sum_{i=1}^{n} i + \frac{6}{n} \sum_{i=1}^{n} 1 \right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2} \cdot \frac{n(n+1)}{2} + \frac{6}{n} \cdot n \right]$$

$$= \lim_{n\to\infty} \left[\frac{9(n+1)(2n+1)}{2n^2} - \frac{9(n+1)}{n} + 6 \right]$$

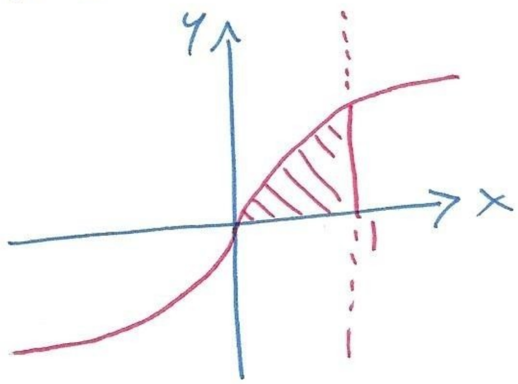
$$= 9 - 9 + 6$$

$$\boxed{= 6}$$

We can generalize the area formula to allow for an irregular partition, where each rectangle or subinterval has its own width Δx_i. Then

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i .$$

eg Find the area under $f(x) = \sqrt[3]{x}$ on $[0,1]$.



What if we use a regular partition?

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i^* = x_i = 0 + \frac{i}{n} = \frac{i}{n}$$

$$f(x_i^*) = \sqrt[3]{\frac{i}{n}}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\frac{i}{n}} \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^{1/3}}{n^{4/3}}$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^{4/3}} \sum_{i=1}^{n} i^{1/3}\right]$$

But we have no summation formula for $\sum_{i=1}^{n} i^{1/3}$

Instead, suppose we choose a partition for which the right endpoint of each subinterval is given by

$$x_i = \frac{i^3}{n^3}$$

Then $f(x_i^*) = f(x_i) = \sqrt[3]{\frac{i^3}{n^3}} = \frac{i}{n}$

Because we are now using an irregular partition, we cannot use the formula $\Delta x = \frac{b-a}{n}$. Instead, we have

$$\Delta x_i = x_i - x_{i-1}$$
$$= \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3}$$
$$= \frac{i^3}{n^3} - \frac{i^3 - 3i^2 + 3i - 1}{n^3}$$
$$= \frac{3i^2 - 3i + 1}{n^3}$$

Now $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{3i^2 - 3i + 1}{n^3}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i^3}{n^4} - \frac{3i^2}{n^4} + \frac{i}{n^4} \right)$$
$$= \lim_{n\to\infty} \left[\frac{3}{n^4} \sum_{i=1}^{n} i^3 - \frac{3}{n^4} \sum_{i=1}^{n} i^2 + \frac{1}{n^4} \sum_{i=1}^{n} i \right]$$
$$= \lim_{n\to\infty} \left[\frac{3}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{3}{n^4} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4} \cdot \frac{n(n+1)}{2} \right]$$
$$= \lim_{n\to\infty} \left[\frac{3(n+1)^2}{4n^2} - \frac{(n+1)(2n+1)}{2n^3} + \frac{n+1}{2n^3} \right]$$
$$= \frac{3}{4} - 0 + 0 \boxed{= \frac{3}{4}}$$

The expression $\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ is called a <u>Riemann sum</u>.