Section 1.4: Integration by Parts

The Product Rule states that

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

This means that

$$\int [f'(x)g(x) + f(x)g'(x)]\,dx = f(x)g(x) + C$$

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x) + C$$

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

Let $v = f(x)$ and $w = g(x)$. Then $dw = f'(x)\,dx$ and $dw = g'(x)\,dx$. Then we have

$$\int g(x)\cdot f'(x)\,dx = f(x)g(x) - \int f(x)\cdot g'(x)\,dx$$

$$\int w\,dv = vw - \int v\,dw$$

which is the <u>integration by parts</u> formula.

eg $\int x \ln(x)\, dx$

One approach to integration by parts is to let

$$w = x \qquad dw = dx$$

$$dv = \ln(x)\,dx \qquad v = \int \ln(x)\,dx$$

which we cannot evaluate

Instead, we will try

$$w = \ln(x) \qquad dw = \frac{1}{x}\,dx$$

$$dv = x\,dx \qquad v = \int x\,dx = \frac{1}{2}x^2$$

By the integration by parts formula, we can write

$$\int x \ln(x)\,dx = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\left[\frac{x^2}{2}\right] + C$$

$$\boxed{= \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C}$$

When choosing dv, we need an expression that we can integrate, ideally an elementary integral. For w, we would like an expression that becomes simpler when it's differentiated, such as x^n (for n a natural number), a logarithmic function or an inverse trigonometric function.

eg $\int xe^x\,dx$

We use integration by parts with

$$w = x \qquad dw = 1\cdot dx = dx$$

$$dv = e^x\,dx \qquad v = e^x$$

so $\int xe^x\,dx = xe^x - \int e^x\,dx$

$$\boxed{= xe^x - e^x + C}$$

Note that v need only be an antiderivative of dv because if we include a constant of integration we would get

$$\int w\,dv = (v+C)w - \int (v+C)\,dw$$

$$= vw + Cw - \int v\,dw - C\int dw$$

$$= vw + Cw - \int v\,dw - Cw$$

$$= vw - \int v\,dw$$

In some cases, integration by parts must be applied several times in order to evaluate a given integral.

eg $\int x^2 e^{6x}\,dx$

We use integration by parts with

$$w = x^2 \qquad dw = 2x\,dx$$
$$dv = e^{6x}\,dx \qquad v = \frac{1}{6}e^{6x}$$

so we have

$$\int x^2 e^{6x}\,dx = \frac{1}{6}x^2 e^{6x} - \int \left(\frac{1}{6}e^{6x}\right)\cdot 2x\,dx$$
$$= \frac{1}{6}x^2 e^{6x} - \frac{1}{3}\int x e^{6x}\,dx$$

Now we use integration by parts again with

$$w = x \qquad dw = 1\cdot dx = dx$$
$$dv = e^{6x}\,dx \qquad v = \frac{1}{6}e^{6x}$$

which gives

$$\int x^2 e^{6x}\,dx = \frac{1}{6}x^2 e^{6x} - \frac{1}{3}\left[\frac{1}{6}x e^{6x} - \int \frac{1}{6}e^{6x}\,dx\right]$$
$$= \frac{1}{6}x^2 e^{6x} - \frac{1}{18}x e^{6x} + \frac{1}{18}\int e^{6x}\,dx$$
$$= \frac{1}{6}x^2 e^{6x} - \frac{1}{18}x e^{6x} + \frac{1}{18}\left[\frac{1}{6}e^{6x}\right] + C$$
$$\boxed{= \frac{1}{6}x^2 e^{6x} - \frac{1}{18}x e^{6x} + \frac{1}{108}e^{6x} + C}$$

eg $\int e^x \cos(x)\,dx$

We use integration by parts with

$$w = \cos(x) \qquad dw = -\sin(x)\,dx$$

$$dv = e^x\,dx \qquad v = e^x$$

so $\int e^x \cos(x)\,dx = e^x\cos(x) - \int e^x \cdot [-\sin(x)\,dx]$

$$= e^x\cos(x) + \int e^x \sin(x)\,dx$$

We use integration by parts again, with

$$w = \sin(x) \qquad dw = \cos(x)\,dx$$

$$dv = e^x\,dx \qquad v = e^x$$

so $\int e^x \cos(x)\,dx = e^x\cos(x) + \left[e^x\sin(x) - \int e^x \cos(x)\,dx\right]$

$$= e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)\,dx$$

$$2\int e^x\cos(x)\,dx = e^x\cos(x) + e^x\sin(x)$$

$$\int e^x\cos(x)\,dx \boxed{= \frac{1}{2}e^x\cos(x) + \frac{1}{2}e^x\sin(x) + C}$$

Sometimes we will apply integration by parts to $\int f(x)\,dx$ by treating as $\int f(x)\cdot 1\,dx$ and setting

$$w = f(x) \quad \text{and} \quad dv = 1\cdot dx = dx$$

eg $\int \ln(x)\,dx$

Let $w = \ln(x)$ $\qquad dw = \frac{1}{x}\,dx$

$dv = dx$ $\qquad v = x$

So, using integration by parts,

$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln(x) - \int dx$$

$$\boxed{= x\ln(x) - x + C}$$

We often need to combine u-substitution and integration by parts.

eg $\int \arctan(x)\,dx$

Let $w = \arctan(x)$ $\qquad dw = \frac{1}{x^2+1}\,dx$

$dv = dx$ $\qquad v = x$

So we have, by integration by parts,

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{x^2+1}\,dx$$

To evaluate the remaining integral, we use u-substitution with

$u = x^2+1$ $\qquad du = 2x\,dx$

$\qquad \frac{1}{2}du = x\,dx$

Now we have

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{1}{u}\cdot\frac{1}{2}\,du$$

$$= x\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x\arctan(x) - \frac{1}{2}\ln|u| + C$$

$$\boxed{= x\arctan(x) - \frac{1}{2}\ln(x^2+1) + C}$$

eg $\int x^3 \cosh(x^2)\, dx$

Let $u = x^2$ so $du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

The integral becomes

$$\int x^3 \cosh(x^2)\, dx = \int x^2 \cosh(x^2) \cdot x\, dx$$

$$= \int u \cosh(u) \cdot \tfrac{1}{2} du$$

$$= \tfrac{1}{2} \int u \cosh(u)\, du$$

Now we apply integration by parts with

$w = u$ $dw = 1 \cdot du = du$

$dv = \cosh(u)\, du$ $v = \sinh(u)$

So we have

$$\int x^3 \cosh(x^2)\, dx = \tfrac{1}{2}\left[u \sinh(u) - \int \sinh(u)\, du\right]$$

$$= \tfrac{1}{2} u \sinh(u) - \tfrac{1}{2} \int \sinh(u)\, du$$

$$= \tfrac{1}{2} u \sinh(u) - \tfrac{1}{2} \cosh(u) + C$$

$$\boxed{= \tfrac{1}{2} x^2 \sinh(x^2) - \tfrac{1}{2} \cosh(x^2) + C}$$

Guidelines for indefinite integration (a work in progress...)

Given an integral $\int f(x)\,dx$, we consider the following in deciding how to evaluate the integral:

① Is this an elementary integral? This includes common integrals, inverse trigonometric integrals, and any combination of them for which the Basic Properties apply, as well as instances of linear composition.

② Can it be rewritten in terms of elementary integrals? We could do this algebraically, using trigonometric identities, using long division, by completing the square, etc.

③ Is u-substitution appropriate?

④ Is integration by parts appropriate?

⑤ Does the integral require a combination of techniques?

eg $\int \sin(\sqrt{x+2})\,dx$

Let $u = \sqrt{x+2}$

$$du = \frac{1}{2}(x+2)^{-1/2}\,dx = \frac{1}{2\sqrt{x+2}}\,dx$$

$$= \frac{1}{2u}\,dx \rightarrow dx = 2u\,du$$

The integral becomes

$$\int \sin(\sqrt{x+2})\,dx = \int \sin(u) \cdot 2u\,du$$
$$= 2\int u\sin(u)\,du$$

Now we use integration by parts with

$$w = u \qquad dw = 1 \cdot du = du$$
$$dv = \sin(u)\,du \qquad v = -\cos(u)$$

So then

$$\int \sin(\sqrt{x+2})\,dx = 2\left[-u\cos(u) + \int \cos(u)\,du\right]$$
$$= 2\left[-u\cos(u) + \sin(u)\right] + C$$
$$\boxed{= -2\sqrt{x+2}\cos(\sqrt{x+2}) + 2\sin(\sqrt{x+2}) + C}$$

In some cases we can use integration by parts to derive a <u>reduction formula</u>, which gives the result of applying this method to integrals of a certain form any number of times.

eg Derive the reduction formula

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$$

where n is any natural number. Use it to evaluate $\int x^3 e^x\,dx$.

We use integration by parts with

$$w = x^n \qquad dw = nx^{n-1}\,dx$$

$$dv = e^x\,dx \qquad v = e^x$$

so

$$\int x^n e^x\,dx = x^n e^x - \int e^x \cdot nx^{n-1}\,dx$$

$$= x^n e^x - n\int x^{n-1} e^x\,dx.$$

Next,

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx$$

$$= x^3 e^x - 3\left[x^2 e^x - 2\int xe^x\,dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int xe^x\,dx$$

$$= x^3 e^x - 3x^2 e^x + 6\left[xe^x - \int e^x\,dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6\int e^x\,dx$$

$$\boxed{= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C}$$