Section 1.3: Integration and Inverse Trigonometric Functions

Recall that $[\arcsin(x)]' = \frac{1}{\sqrt{1-x^2}}$

$$[\arccos(x)]' = \frac{-1}{\sqrt{1-x^2}}.$$

So we can write

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

and $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$

However, we could instead write

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= -\arcsin(x) + C$$

Thus we generally only use $\arcsin(x)$ in our integration results not $\arccos(x)$. The same is true for $\arctan(x)$ and $\operatorname{arcsec}(x)$.

We now have $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$.

More generally, we have the following:

<u>Theorem</u>: $\displaystyle\int \frac{1}{\sqrt{k^2-x^2}}\,dx = \arcsin\left(\frac{x}{k}\right) + C$

for any constant $k > 0$

<u>Proof</u>: We can rewrite the integral as

$$\int \frac{1}{\sqrt{k^2-x^2}}\,dx = \int \frac{1}{\sqrt{k^2\left(1-\frac{x^2}{k^2}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{k^2}\cdot\sqrt{1-\frac{x^2}{k^2}}}\,dx$$

$$= \int \frac{1}{k\sqrt{1-\frac{x^2}{k^2}}}\,dx$$

$$= \int \frac{1}{k\sqrt{1-\left(\frac{x}{k}\right)^2}}\,dx$$

Let $u = \frac{x}{k}$ so $du = \frac{1}{k}\,dx$ and the integral becomes

$$\int \frac{1}{\sqrt{k^2-x^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \arcsin(u) + C$$

$$= \arcsin\left(\frac{x}{k}\right) + C.$$

eg $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{\sqrt{5^2-x^2}}\,dx$

This is an arcsine integral with $k=5$. Thus

$$\int \frac{1}{\sqrt{25-x^2}}\,dx \boxed{= \arcsin\left(\frac{x}{5}\right) + C}.$$

Theorem: $\displaystyle\int \frac{1}{x^2+k^2}\,dx = \frac{1}{k}\arctan\left(\frac{x}{k}\right) + C$

eg $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{e^x}{(e^x)^2+1}\,dx$

Let $u = e^x$ then $du = e^x\,dx$ so the integral becomes

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{1}{u^2+1}\,du$$

which is an arctangent integral with $k=1$. Hence

$$\int \frac{e^x}{e^{2x}+1}\,dx = \frac{1}{1}\arctan\left(\frac{u}{1}\right) + C$$

$$\boxed{= \arctan(e^x) + C}$$

Theorem: $\displaystyle\int \frac{1}{x\sqrt{x^2-k^2}}\,dx = \frac{1}{k}\,\mathrm{arcsec}\left(\frac{x}{k}\right) + C$

eg $\int \frac{1}{x\sqrt{16x^2-9}}\,dx$

$$= \int \frac{1}{x\sqrt{16\left(x^2-\frac{9}{16}\right)}}\,dx$$

$$= \int \frac{1}{x \cdot 4\sqrt{x^2-\frac{9}{16}}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{x\sqrt{x^2-\left(\frac{3}{4}\right)^2}}\,dx$$

This is an arcsecant integral with $k = 3/4$

$$= \frac{1}{4}\cdot\frac{1}{3/4}\,\text{arcsec}\left(\frac{x}{3/4}\right) + C$$

$$\boxed{= \frac{1}{3}\,\text{arcsec}\left(\frac{4x}{3}\right) + C}$$

Alternatively we can rewrite the integral as

$$\int \frac{1}{x\sqrt{(4x)^2-9}}\,dx$$

Let $u = 4x$

$du = 4\,dx \rightarrow \frac{1}{4}\,du = dx$

$x = \frac{1}{4}u$

$$= \int \frac{1}{\frac{1}{4}u\sqrt{u^2-9}}\cdot\frac{1}{4}\,du$$

$$= \int \frac{1}{u\sqrt{u^2-9}}\,du$$

This is an arcsecant integral with $k = 3$

$$= \frac{1}{3}\,\text{arcsec}\left(\frac{u}{3}\right) + C$$

$$\boxed{= \frac{1}{3}\,\text{arcsec}\left(\frac{4x}{3}\right) + C}$$

Integrals of rational functions often lead to arctangent functions.

eg $$\int \frac{x+3}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx$$

For the first integral, we let
$$u = x^2+9$$
$$du = 2x\,dx$$
$$\tfrac{1}{2}\,du = x\,dx$$

Thus
$$\int \frac{x}{x^2+9}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du$$
$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln|x^2+9| + C$$
$$= \frac{1}{2}\ln(x^2+9) + C$$

For the second integral, this is an arctangent form with $k=3$ so
$$\int \frac{3}{x^2+9}\,dx = 3\int \frac{1}{x^2+9}\,dx$$
$$= 3\cdot\frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$
$$= \arctan\left(\frac{x}{3}\right) + C$$

Finally, $$\int \frac{x+3}{x^2+9}\,dx \boxed{= \frac{1}{2}\ln(x^2+9) + \arctan\left(\frac{x}{3}\right) + C}$$

In order to apply an inverse trigonometric integral to a case where the quadratic expression in the denominator includes a linear term, we must first <u>complete the square</u>.

Suppose we have $x^2 + bx + c$ where b and c are constants. Then we first rewrite this as

$$\begin{aligned}(x^2 + bx) + c &= \left(x^2 + bx + \left(\tfrac{b}{2}\right)^2\right) + \left(c - \left(\tfrac{b}{2}\right)^2\right) \\ &= \left(x + \tfrac{b}{2}\right)^2 + \left(c - \left(\tfrac{b}{2}\right)^2\right)\end{aligned}$$

eg $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x-4}}\,dx$

We start by completing the square:

$$\begin{aligned}x^2 - 2x - 4 &= (x^2 - 2x) - 4 \\ &= (x^2 - 2x + 1) - 4 - 1 \\ &= (x-1)^2 - 5\end{aligned}$$

The integral becomes

$$\int \frac{1}{(x-1)\sqrt{x^2-2x-4}}\,dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-5}}\,dx$$

$$\int \frac{1}{(x-1)\sqrt{x^2-2x-4}}\,dx$$

$$= \int \frac{1}{(x-1)\sqrt{(x-1)^2-5}}\,dx$$

Let $u = x-1$

$du = 1 \cdot dx = dx$

The integral becomes

$$\int \frac{1}{(x-1)\sqrt{x^2-2x-4}}\,dx = \int \frac{1}{u\sqrt{u^2-5}}\,du$$

$$= \frac{1}{\sqrt{5}} \operatorname{arcsec}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$\boxed{= \frac{\sqrt{5}}{5} \operatorname{arcsec}\left(\frac{\sqrt{5}}{5}(x-1)\right) + C}$$

To complete the square for an expression of the form

$$ax^2+bx+c$$

where $a \neq 1$, we first factor a outside the entire trinomial. Then we complete the square for the remaining quadratic expression, and multiply a back in.

eg $\displaystyle\int \frac{1}{4x^2+12x+10}\,dx$

We complete the square:

$$\begin{aligned}
4x^2+12x+10 &= 4\left[x^2+3x+\tfrac{5}{2}\right] \\
&= 4\left[(x^2+3x)+\tfrac{5}{2}\right] \\
&= 4\left[\left(x^2+3x+\tfrac{9}{4}\right)+\tfrac{5}{2}-\tfrac{9}{4}\right] \\
&= 4\left[\left(x+\tfrac{3}{2}\right)^2+\tfrac{1}{4}\right] \\
&= 4\left(x+\tfrac{3}{2}\right)^2+1 \\
&= 2^2\left(x+\tfrac{3}{2}\right)^2+1 \\
&= (2x+3)^2+1
\end{aligned}$$

The integral can be rewritten as

$$\int \frac{1}{4x^2+12x+10}\,dx = \int \frac{1}{(2x+3)^2+1}\,dx$$

Let $u = 2x+3$
$du = 2dx$
$\frac{1}{2}du = dx$

$$\begin{aligned}
&= \int \frac{1}{u^2+1}\cdot \frac{1}{2}\,du \\
&= \frac{1}{2}\int \frac{1}{u^2+1}\,du \\
&= \frac{1}{2}\cdot\frac{1}{1}\arctan\left(\frac{u}{1}\right)+C \\
&\boxed{= \frac{1}{2}\arctan(2x+3)+C}
\end{aligned}$$

eg $\int \frac{1}{\sqrt{20+8x-x^2}}\,dx$

We write $20+8x-x^2 = -\left[x^2-8x-20\right]$

$$= -\left[(x^2-8x)-20\right]$$

$$= -\left[(x^2-8x+16)-20-16\right]$$

$$= -\left[(x-4)^2-36\right]$$

$$= 36-(x-4)^2$$

Now the integral becomes

$$\int \frac{1}{\sqrt{20+8x-x^2}}\,dx = \int \frac{1}{\sqrt{36-(x-4)^2}}\,dx \qquad \text{Let } u = x-4, \quad du = dx$$

$$= \int \frac{1}{\sqrt{36-u^2}}\,du$$

$$= \arcsin\left(\frac{u}{6}\right) + C$$

$$\boxed{= \arcsin\left(\frac{x-4}{6}\right) + C}$$

Very rarely, the process of integration will naturally lead to an arccosine, arccotangent or arccosecant function, usually when one or more is present in the integrand.

eg $\int \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx$

Let $u = \arccos(x)$ so $du = \frac{-1}{\sqrt{1-x^2}}\,dx$

$$-du = \frac{1}{\sqrt{1-x^2}}\,dx$$

The integral becomes

$$\int \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx = \int u \cdot (-du)$$

$$= -\int u\,du$$

$$= -\left[\frac{u^2}{2}\right] + C$$

$$\boxed{= -\frac{1}{2}\arccos^2(x) + C}$$