Section 1.1: Indefinite Integration

Def'n: An antiderivative of the function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$ for all x at which $f(x)$ is defined.
The process of finding an antiderivative is called antidifferentiation.

eg We know that $[x^2]' = 2x$ so x^2 is an antiderivative of $2x$.
But $[x^2+1]' = 2x$ as well, so x^2+1 is also an antiderivative of $2x$.
Likewise, $[x^2 - \sqrt{3}]' = 2x$ so $x^2 - \sqrt{3}$ is another antiderivative of $2x$.
In fact, $x^2 + C$ for any real constant C is an antiderivative of $2x$. We call this "most general" antiderivative the indefinite integral of $2x$.

Def'n: If $f(x)$ is a function with antiderivative $F(x)$ then the indefinite integral of $f(x)$ is $F(x) + C$ where C is an arbitrary real constant known as the constant of integration. If $f(x)$ possesses an indefinite integral then it is integrable. The process of finding the indefinite integral is indefinite integration.

eg We know that $[\sin(x)]' = \cos(x)$ so the indefinite integral of $\cos(x)$ is $\sin(x) + C$.

We denote the indefinite integral of $f(x)$ as

$$\int f(x)\,dx$$

where $\int$ is the integral symbol and dx is called a differential. The function $f(x)$ being integrated is the integrand.

eg $\int 2x\,dx = x^2 + C$

$\int \cos(x)\,dx = \sin(x) + C$

The general process for indefinite integration is as follows:

1. Rewrite the integrand in a more convenient form (if necessary)
2. Identify the indefinite integral based on our knowledge of derivatives
3. Perform any obvious simplifications (if possible)

eg $\int \frac{5x^{9/2}}{\sqrt{x}}\, dx$

We can rewrite the integrand:

$$\frac{5x^{9/2}}{\sqrt{x}} = \frac{5x^{9/2}}{x^{1/2}} = 5x^4$$

So, from the Power Rule for derivatives,

$$\int 5x^4\, dx \boxed{= x^5 + C}$$

We can use our results for derivatives to establish several <u>common integrals</u>:

1. $\int 0\, dx = C$
2. $\int 1\, dx = \int dx = x + C$

The Power Rule for derivatives states that

$$[x^r]' = rx^{r-1}$$

so $\int rx^{r-1}\,dx = x^r + C$.

Likewise, $[x^{r+1}]' = (r+1)\,x^r$

$$\left[\frac{x^{r+1}}{r+1}\right]' = \frac{(r+1)\,x^r}{r+1} = x^r$$

So now we have:

3 $\displaystyle\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$ for $r \neq -1$

which is the <u>Power Rule for integrals</u>

eg $\displaystyle\int x^7\,dx \boxed{= \frac{x^8}{8} + C = \frac{1}{8}x^8 + C}$

eg $\displaystyle\int \sqrt{\frac{1}{x^3}}\,dx = \int \frac{1}{x^{3/2}}\,dx = \int x^{-3/2}\,dx$

$$= \frac{x^{-1/2}}{-1/2} + C$$

$$\boxed{\begin{aligned} &= -2x^{-1/2} + C \\ &= \frac{-2}{\sqrt{x}} + C \end{aligned}}$$

What about $\int x^{-1}\,dx = \int \frac{1}{x}\,dx$?

We know that $[\ln(x)]' = \frac{1}{x}$ but $\ln(x)$ is defined only for $x>0$ while $\frac{1}{x}$ is defined for all $x \neq 0$.

But observe that $[\ln(-x)]' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$.

Thus $\int \frac{1}{x}\,dx = \ln(x) + C$ for $x>0$ and $\int \frac{1}{x}\,dx = \ln(-x) + C$ for $x<0$ or, put together:

(4) $\int \frac{1}{x}\,dx = \ln|x| + C$

Common integrals:

(1) $\int 0\,dx = C$

(2) $\int 1\,dx = \int dx = x + C$

(3) $\int x^r\,dx = \dfrac{x^{r+1}}{r+1} + C \quad , \quad r \neq -1$

(4) $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$

(5) $\int e^x\,dx = e^x + C$

Recall that $[b^x]' = b^x \ln(b)$ for $b > 0$, $b \neq 1$

$$\left[\frac{b^x}{\ln(b)}\right]' = \frac{b^x \ln(b)}{\ln(b)} = b^x$$

(6) $\int b^x\,dx = \dfrac{b^x}{\ln(b)} + C$

(7) $\int \cos(x)\,dx = \sin(x) + C$

(8) $\int \sin(x)\,dx = -\cos(x) + C$

(9) $\int \sec(x)\tan(x)\,dx = \sec(x) + C$

(10) $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

(11) $\int \sec^2(x)\,dx = \tan(x) + C$

(12) $\int \csc^2(x)\,dx = -\cot(x) + C$

(13) $\int \cosh(x)\,dx = \sinh(x) + C$

(14) $\int \sinh(x)\,dx = \cosh(x) + C$

Theorem: Basic Properties of Indefinite Integration

If $f(x)$ and $g(x)$ are integrable functions then

(1) $\int k f(x)\,dx = k \int f(x)\,dx$ for any constant k

(2) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(3) $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

eg $\int (x^5 - 4x^2 + 6)\,dx$

$= \int x^5\,dx - 4\int x^2\,dx + 6\int dx$

$= \left[\frac{x^6}{6} + C_1\right] - 4\left[\frac{x^3}{3} + C_2\right] + 6\left[x + C_3\right]$

$= \frac{1}{6}x^6 - \frac{4}{3}x^3 + 6x + C_1 - 4C_2 + 6C_3$

$\boxed{= \frac{1}{6}x^6 - \frac{4}{3}x^3 + 6x + C}$

eg $\int \left[\pi \sin(x) - \frac{1}{2}\cos(x)\right] dx$

$= \pi \int \sin(x)\,dx - \frac{1}{2}\int \cos(x)\,dx$

$= \pi\left[-\cos(x)\right] - \frac{1}{2}\left[\sin(x)\right] + C$

$\boxed{= -\pi\cos(x) - \frac{1}{2}\sin(x) + C}$

eg $\int \left(\frac{x^2}{3} - \frac{4}{5x^3} + \frac{3}{x} \right) dx$

$$= \frac{1}{3} \int x^2 \, dx - \frac{4}{5} \int x^{-3} \, dx + 3 \int \frac{1}{x} \, dx$$

$$= \frac{1}{3} \left[\frac{x^3}{3} \right] - \frac{4}{5} \left[\frac{x^{-2}}{-2} \right] + 3 \cdot \ln|x| + C$$

$$\boxed{= \frac{1}{9} x^3 + \frac{2}{5} x^{-2} + 3 \ln|x| + C}$$

Unfortunately, there is no general method for integrating products, quotients or composite functions.

However some integrals involving these operations can be evaluated by first rewriting the integrand so that it is just a sum, difference or constant multiple of the common integrals.

eg $\int (\sqrt{x} - 1)^2 \, dx$

$$= \int (\sqrt{x} - 1)(\sqrt{x} - 1) \, dx$$

$$= \int (x - 2\sqrt{x} + 1) \, dx$$

$$= \int x \, dx - 2 \int x^{1/2} \, dx + \int dx$$

$$= \frac{1}{2} x^2 - 2 \left[\frac{x^{3/2}}{3/2} \right] + x + C$$

$$\boxed{= \frac{1}{2} x^2 - \frac{4}{3} x^{3/2} + x + C}$$

eg $\displaystyle\int \frac{\sin(\theta)}{\cos^2(\theta)}\, d\theta$

$$= \int \frac{1}{\cos(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)}\, d\theta$$

$$= \int \sec(\theta)\tan(\theta)\, d\theta \boxed{= \sec(\theta) + C}$$

Now consider the linear composition of a simple function and explore how this affects its integral. In general, this has the form

$$\int f(mx+b)\, dx$$

where $\int f(x)\,dx = F(x) + C$ is known, and m and b are constants where $m \neq 0$.

eg $\int (2x-3)^{60}\,dx$

Compare: $\int x^{60}\,dx = \frac{x^{61}}{61} + C$

Guess: $\int (2x-3)^{60}\,dx = \frac{(2x-3)^{61}}{61} + C$

Check by differentiating:

$$\left[\frac{(2x-3)^{61}}{61} + C\right]' = \frac{1}{61}\cdot 61(2x-3)^{60}\cdot 2 + 0$$
$$= 2(2x-3)^{60}$$
$$\neq (2x-3)^{60}$$

So our guess is <u>incorrect</u>. However, we can write

$$\int 2(2x-3)^{60}\,dx = \frac{(2x-3)^{61}}{61} + C$$

$$2\int (2x-3)^{60}\,dx = \frac{(2x-3)^{61}}{61} + C$$

$$\int (2x-3)^{60}\,dx = \frac{(2x-3)^{61}}{2\cdot 61} + C$$

$$\boxed{= \frac{(2x-3)^{61}}{122} + C}$$

eg $\int \sin(7x)\,dx$

Compare: $\int \sin(x)\,dx = -\cos(x) + C$

Guess: $\int \sin(7x)\,dx = -\cos(7x) + C$

Check: $[-\cos(7x) + C]' = \sin(7x) \cdot 7 + 0$

$= 7\sin(7x)$

Thus $\int 7\sin(7x)\,dx = -\cos(7x) + C$

$$\boxed{\int \sin(7x)\,dx = -\frac{1}{7}\cos(7x) + C}$$

Theorem: If $f(x)$ is a function with antiderivative $F(x)$ then

$$\int f(mx+b)\,dx = \frac{F(mx+b)}{m} + C.$$

eg $\int \frac{1}{(2-x)^6}\,dx$ Compare: $\int \frac{1}{x^6}\,dx$

$= \int x^{-6}\,dx = \frac{x^{-5}}{-5} + C$

Then $\int \frac{1}{(2-x)^6}\,dx = \frac{(2-x)^{-5}}{(-5)\cdot(-1)} + C$

$$\boxed{\begin{aligned} &= \frac{1}{5}(2-x)^{-5} + C \\ &= \frac{1}{5(2-x)^5} + C \end{aligned}}$$

eg $\int \sec^2\left(\frac{4x+6}{3}\right) dx$

Compare: $\int \sec^2(x)\,dx$
$= \tan(x) + C$

$= \int \sec^2\left(\frac{4}{3}x+2\right) dx$

$= \dfrac{\tan\left(\frac{4}{3}x+2\right)}{4/3} + C$

$\boxed{= \frac{3}{4}\tan\left(\frac{4}{3}x+2\right) + C}$

eg $\int \dfrac{1}{\frac{1}{4}x-3}\, dx$

Compare: $\int \frac{1}{x}\, dx$
$= \ln|x| + C$

$= \dfrac{\ln\left|\frac{1}{4}x-3\right|}{1/4} + C$

$\boxed{= 4\ln\left|\frac{1}{4}x-3\right| + C}$

Alternatively, we could first rewrite:

$$\int \frac{1}{\frac{1}{4}x-3}\, dx = \int \frac{4}{x-12}\, dx$$

$$= 4\int \frac{1}{x-12}\, dx$$

$$= 4\,\frac{\ln|x-12|}{1} + C$$

$$\boxed{= 4\ln|x-12| + C}$$

But we can write

$$4\ln\left|\tfrac{1}{4}x-3\right|+C = 4\ln\left|\tfrac{1}{4}(x-12)\right|+C$$

$$= 4\left(\ln\left|\tfrac{1}{4}\right|+\ln|x-12|\right)+C$$

$$= 4\ln|x-12| + 4\ln\left|\tfrac{1}{4}\right| + C$$

$$= 4\ln|x-12| + C$$

Could we obtain a similar result for quadratic composition instead of linear composition? That is, for an integral of the form

$$\int f(ax^2+bx+c)\,dx$$

eg $\int (2x^2-3)^{60}\,dx$ Compare: $\int x^{60}\,dx = \frac{x^{61}}{61}+C$

Guess: $\int (2x^2-3)^{60}\,dx = \frac{(2x^2-3)^{61}}{61}+C$

Check: $\left[\frac{(2x^2-3)^{61}}{61}+C\right]' = \frac{61(2x^2-3)^{60}\cdot 4x}{61}+0$

$$= 4x(2x^2-3)^{60}$$

Thus $\int 4x(2x^2-3)^{60}\,dx = \frac{(2x^2-3)^{61}}{61}+C$

but we have no way to obtain $\int (2x^2-3)^{60}\,dx$.

Sadly, no simple pattern exists beyond linear composition.

Note that

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

eg $\int 2x\,dx = x^2 + C$

$$\frac{d}{dx}\left[\int 2x\,dx\right] = \frac{d}{dx}\left[x^2 + C\right] = 2x + 0 \boxed{= 2x}$$

However,

$$\int f'(x)\,dx = f(x) + C.$$

eg $[x^2]' = 2x$

$$\int [x^2]'\,dx = \int 2x\,dx \boxed{= x^2 + C}$$