MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 **Mathematics 1001** WINTER 2024

SOLUTIONS

[5] 1. (a) We can write

$$\frac{2x^2-4x-1}{3x^3+x^2+3x+1} = \frac{2x^2-4x-1}{(3x+1)(x^2+1)} = \frac{A}{3x+1} + \frac{Bx+D}{x^2+1}.$$

Thus

$$2x^2-4x-1 = A(x^2+1) + (Bx+D)(3x+1).$$

When $x = -\frac{1}{3}$, we have $\frac{5}{9} = A \cdot \frac{10}{9}$ so $A = \frac{1}{2}$. When $x = 0$, we have $-1 = A + D$ so $D = -1 - A = -\frac{3}{2}$. Finally, when (say) $x = 1$, we have $-3 = 2A + 4B + 4D = 1 + 4B - 6$ so $B = \frac{1}{2}$. Hence

$$\begin{aligned}
\int \frac{2x^2-4x-1}{3x^3+x^2+3x+1} &= \int \left(\frac{\frac{1}{2}}{3x+1} + \frac{\frac{1}{2}x - \frac{3}{2}}{x^2+1} \right) dx \\
&= \frac{1}{2}\int \frac{1}{3x+1}\,dx + \frac{1}{2}\int \frac{x}{x^2+1}\,dx - \frac{3}{2}\int \frac{1}{x^2+1}\,dx \\
&= \frac{1}{6}\ln|3x+1| + \frac{1}{2}\int \frac{x}{x^2+1}\,dx - \frac{3}{2}\arctan(x).
\end{aligned}$$

For the remaining integral, we let $u = x^2+1$ so $\frac{1}{2}\,du = x\,dx$. Then

$$\begin{aligned}
\int \frac{2x^2-4x-1}{3x^3+x^2+3x+1} &= \frac{1}{6}\ln|3x+1| + \frac{1}{4}\int \frac{1}{u}\,du - \frac{3}{2}\arctan(x) \\
&= \frac{1}{6}\ln|3x+1| + \frac{1}{4}\ln|u| - \frac{3}{2}\arctan(x) + C \\
&= \frac{1}{6}\ln|3x+1| + \frac{1}{4}\ln(x^2+1) - \frac{3}{2}\arctan(x) + C.
\end{aligned}$$

[5] (b) We write

$$\begin{aligned}
\int \frac{\cos^5(3x)}{\sin^4(3x)}\,dx &= \int \frac{\cos^4(3x)}{\sin^4(3x)} \cdot \cos(3x)\,dx \\
&= \int \frac{[1-\sin^2(3x)]^2}{\sin^4(3x)} \cdot \cos(3x)\,dx.
\end{aligned}$$

Now we let $u = \sin(3x)$ so $du = 3\cos(3x)\,dx$ and $\frac{1}{3}\,du = \cos(3x)\,dx$. The integral becomes

$$\begin{aligned}\int \frac{\cos^5(3x)}{\sin^4(3x)}\,dx &= \frac{1}{3}\int \frac{[1-u^2]^2}{u^4}\,du \\ &= \frac{1}{3}\int \frac{1-2u^2+u^4}{u^4}\,du \\ &= \frac{1}{3}\int (u^{-4}-2u^{-2}+1)\,du \\ &= \frac{1}{3}\left[\frac{u^{-3}}{-3} - 2\cdot\frac{u^{-1}}{-1} + u\right] + C \\ &= -\frac{1}{9\sin^3(3x)} + \frac{2}{3\sin(3x)} + \frac{1}{3}\sin(3x) + C.\end{aligned}$$

[5] (c) Let $x = 5\sec(\theta)$ so $dx = 5\sec(\theta)\tan(\theta)\,d\theta$. Then $x^3 = 125\sec^3(\theta)$ and

$$\sqrt{x^2-25} = \sqrt{25\sec^2(\theta)-25} = 5\tan(\theta).$$

The integral becomes

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x^3}\,dx &= \int \frac{5\tan(\theta)}{125\sec^3(\theta)}\cdot 5\sec(\theta)\tan(\theta)\,d\theta \\ &= \frac{1}{5}\int \frac{\tan^2(\theta)}{\sec^2(\theta)}\,d\theta \\ &= \frac{1}{5}\int \sin^2(\theta)\,d\theta \\ &= \frac{1}{5}\int \left[\frac{1-\cos(2\theta)}{2}\right]\,d\theta \\ &= \frac{1}{10}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C \\ &= \frac{1}{10}\theta - \frac{1}{10}\sin(\theta)\cos(\theta) + C.\end{aligned}$$

Since $\sec(\theta) = \frac{x}{5}$, $\theta = \operatorname{arcsec}\left(\frac{x}{5}\right)$ and $\cos(\theta) = \frac{5}{x}$. Furthermore, we can draw a right triangle with interior angle θ, adjacent sidelength 5 and hypotenuse of length x. By the Pythagorean theorem, the opposite sidelength is $\sqrt{x^2-25}$ and so

$$\sin(\theta) = \frac{\sqrt{x^2-25}}{x}.$$

Finally, then,

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x^3}\,dx &= \frac{1}{10}\operatorname{arcsec}\left(\frac{x}{5}\right) - \frac{1}{10}\cdot\frac{\sqrt{x^2-25}}{x}\cdot\frac{5}{x} + C \\ &= \frac{1}{10}\operatorname{arcsec}\left(\frac{x}{5}\right) - \frac{\sqrt{x^2-25}}{2x^2} + C.\end{aligned}$$

[5] (d) Let $x = 4\sin(\theta)$ so $dx = 4\cos(\theta)\,d\theta$. Then

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2(\theta)} = \sqrt{16[1-\sin^2(\theta)]} = \sqrt{16\cos^2(\theta)} = 4\cos(\theta).$$

When $x = 0$, we have $0 = 4\sin(\theta)$ so $\theta = \arcsin(0) = 0$. When $x = 2\sqrt{2}$, we have $2\sqrt{2} = 4\sin(\theta)$ so $\theta = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. Hence the integral becomes

$$\begin{aligned}
\int_0^{2\sqrt{2}} \frac{x}{\sqrt{16-x^2}}\,dx &= \int_0^{\frac{\pi}{4}} \frac{4\sin(\theta)}{4\cos(\theta)} \cdot 4\cos(\theta)\,d\theta \\
&= 4\int_0^{\frac{\pi}{4}} \sin(\theta)\,d\theta \\
&= 4\Big[-\cos(\theta)\Big]_0^{\frac{\pi}{4}} \\
&= 4\left[-\cos\left(\frac{\pi}{4}\right) + \cos(0)\right] \\
&= 4\left(-\frac{\sqrt{2}}{2} + 1\right) \\
&= 4 - 2\sqrt{2}.
\end{aligned}$$

[5] 2. We use integration by parts, with $w = x^2$ so $dw = 2x\,dx$, and $dv = e^{4x}\,dx$ so $v = \frac{1}{4}e^{4x}$. Then

$$\begin{aligned}
\int_{-\infty}^0 x^2 e^{4x}\,dx &= \lim_{T\to-\infty} \int_T^0 x^2 e^{4x}\,dx \\
&= \lim_{T\to-\infty} \left(\left[\frac{1}{4}x^2 e^{4x}\right]_T^0 - \frac{1}{2}\int_T^0 x e^{4x}\,dx \right).
\end{aligned}$$

We use integration by parts again, with $w = x$ so $dw = dx$, and $dv = e^{4x}\,dx$ so $v = \frac{1}{4}e^{4x}$. Now

$$\begin{aligned}
\int_{-\infty}^0 x^2 e^{4x}\,dx &= \lim_{T\to-\infty} \left(\left[\frac{1}{4}x^2 e^{4x} - \frac{1}{8}x e^{4x}\right]_T^0 + \frac{1}{8}\int_T^0 e^{4x}\,dx \right) \\
&= \lim_{T\to-\infty} \left[\frac{1}{4}x^2 e^{4x} - \frac{1}{8}x e^{4x} + \frac{1}{32}e^{4x}\right]_T^0 \\
&= \lim_{T\to-\infty} \left[\frac{1}{32} - \frac{1}{4}T^2 e^{4T} + \frac{1}{8}T e^{4T} - \frac{1}{32}e^{4T}\right].
\end{aligned}$$

First note that

$$\lim_{T\to-\infty} e^{4T} = 0.$$

Next,

$$\lim_{T\to-\infty} T e^{4T} = \lim_{T\to-\infty} \frac{T}{e^{-4T}} \overset{\text{H}}{=} \lim_{T\to-\infty} \frac{1}{-4e^{-4T}} = -\frac{1}{4}\lim_{T\to-\infty} e^{4T} = 0.$$

Lastly,

$$\lim_{T\to-\infty} T^2e^{4T} = \lim_{T\to-\infty} \frac{T^2}{e^{-4T}} \overset{\text{H}}{=} \lim_{T\to-\infty} \frac{2T}{-4e^{-4T}} = -\frac{1}{2}\lim_{T\to-\infty} \frac{T}{e^{-4T}}.$$

(Here we can use l'Hôpital's Rule a second time, or just apply our previous result.) So, finally,

$$\int_{-\infty}^{0} x^2e^{4x}\,dx = \frac{1}{32} - 0 + 0 - 0 = \frac{1}{32}.$$