MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 | **Mathematics 1001** | WINTER 2024

SOLUTIONS

[2] 1. (a) Observe that $y - x + 1 = 0$ can be rewritten as $y = x - 1$, so it is a line moving upwards to the right with intercepts $(1, 0)$ and $(0, -1)$. The curve $x + y^2 - 3 = 0$ could be written as $x = 3 - y^2$. This is a parabola opening to the left (because the coefficient of y^2 is negative). Its intercepts are $\left(0, \sqrt{3}\right)$, $\left(0, -\sqrt{3}\right)$ and $(3, 0)$, the last of which is also the vertex of the parabola. To find any points of intersection, we substitute $y = x - 1$ into the quadratic equation and obtain

$$\begin{aligned} x + (x-1)^2 - 3 &= 0 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0, \end{aligned}$$

so $x = 2$ and $x = -1$. Thus we can sketch the curve, as found in Figure 1.

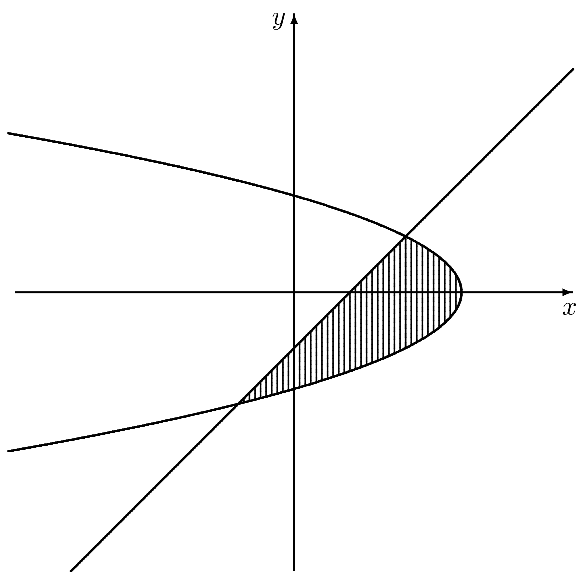

Figure 1: Question 2(a)

[4] (b) From the graph it's clear that the region is not vertically simple, because the top boundary curve is sometimes the line and sometimes the upper branch of the parabola. Thus, in order to integrate with respect to x, we need to split up the region into two vertically simple regions.

Between the intersection points $x = -1$ and $x = 2$, the top boundary curve is the line $y = x - 1$, while the bottom boundary curve is the bottom part of the parabola. To express it as a function of x, we solve $x + y^2 - 3 = 0$ for y, and find

$$y = \sqrt{3 - x} \quad \text{or} \quad y = -\sqrt{3 - x}.$$

The second result must represent the bottom part of the parabola. Now the area of this part of the region is given by

$$\begin{aligned} A &= \int_{-1}^{2} \left[(x-1) - \left(-\sqrt{3-x}\right)\right] \, dx \\ &= \int_{-1}^{2} \left[x - 1 + \sqrt{3-x}\right] \, dx \\ &= \left[\frac{1}{2}x^2 - x - \frac{2}{3}(3-x)^{\frac{3}{2}}\right]_{-1}^{2} \\ &= \frac{19}{6}. \end{aligned}$$

Now, between $x = 2$ and the vertex $x = 3$, the top boundary curve is the top part of the parabola, while the bottom boundary curve is still the bottom part of the parabola. Thus the area of this part of the region is given by

$$A = \int_{2}^{3} \left[\sqrt{3-x} - \left(-\sqrt{3-x}\right)\right] \, dx = 2\int_{2}^{3} \sqrt{3-x}\, dx = -\frac{4}{3}\left[(3-x)^{\frac{3}{2}}\right]_{2}^{3} = \frac{4}{3}.$$

Finally, the total area of the region must be

$$A = \frac{19}{6} + \frac{4}{3} = \frac{9}{2}.$$

[4] (c) This problem is much more easily solved by integrating with respect to y. Now the rightmost boundary curve is $x = 3 - y^2$ and the leftmost boundary curve is $x = y + 1$. Since the x-coordinates of the points of the intersection are $x = 2$ and $x = -1$, the corresponding y-coordinates are $y = 1$ and $y = -2$ (as can be found by substituting into either equation). Thus the area of the whole region is

$$A = \int_{-2}^{1} [(3-y^2) - (y+1)]\, dy = \int_{-2}^{1} [-y^2 - y + 2]\, dy = \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y\right]_{-2}^{1} = \frac{9}{2}.$$

[5] 2. (a) The graph can be found in Figure 2. We need to find the points of intersection, so we set

$$x^2 - 4 = 4 - 3x^2 \implies 4x^2 = 8 \implies x^2 = 2$$

so $x = \pm\sqrt{2}$. On the interval $[-\sqrt{2}, \sqrt{2}]$, we can see that $y = 4 - 3x^2$ is the upper boundary curve, and $y = x^2 - 4$ is the lower boundary curve. Therefore the area of the region is

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} [(4 - 3x^2) - (x^2 - 4)]\, dx$$
$$= \int_{-\sqrt{2}}^{\sqrt{2}} [8 - 4x^2]\, dx$$
$$= \left[8x - \frac{4}{3}x^3\right]_{-\sqrt{2}}^{\sqrt{2}}$$
$$= \frac{32\sqrt{2}}{3}.$$

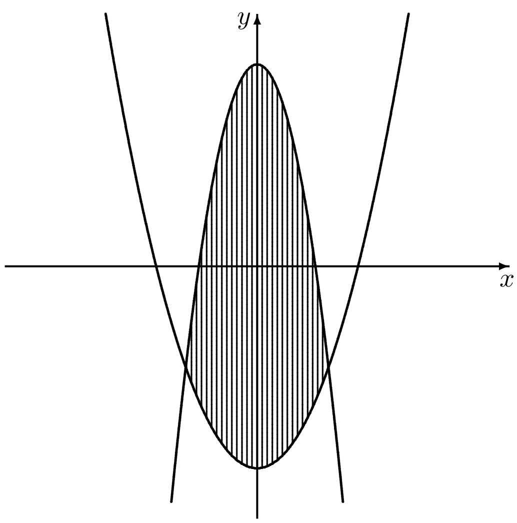

Figure 2: Question 3(a)

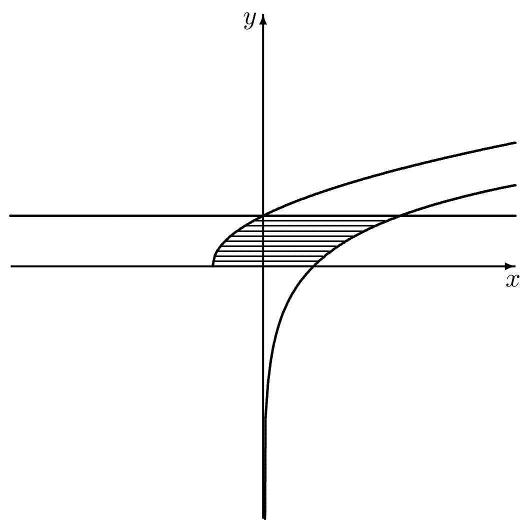

Figure 3: Question 3(b)

[5] (b) The graph can be found in Figure 3. Here we can see that the two boundary curves never intersect, and that the region is horizontally simple; thus we should work in terms of functions of y. As such, the function $y = \sqrt{x+1}$ becomes $x = y^2 - 1$, while the function $y = \ln(x)$ becomes $x = e^y$. From the graph, it is clear that $e^y \geq (y^2 - 1)$ on the interval from $y = 0$ (the x-axis) to $y = 1$. Thus

$$A = \int_0^1 [e^y - (y^2 - 1)]\, dy$$
$$= \int_0^1 [e^y - y^2 + 1]\, dy$$
$$= \left[e^y - \frac{1}{3}y^3 + y\right]_0^1$$
$$= e - \frac{1}{3}.$$