MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 4	**MATHEMATICS 1001**	Winter 2024

SOLUTIONS

[5] 1. (a) We form a regular partition where

$$\Delta x = \frac{4-(-2)}{n} = \frac{6}{n}$$

and choose the sample point

$$x_i^* = -2 + i \cdot \frac{6}{n} = \frac{6i}{n} - 2.$$

Thus

$$f(x_i^*) = \left(\frac{6i}{n} - 2\right)^2 - 3\left(\frac{6i}{n} - 2\right) + 6 = \frac{36i^2}{n^2} - \frac{42i}{n} + 16.$$

Now we have

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{36i^2}{n^2} - \frac{42i}{n} + 16\right] \cdot \frac{6}{n} \\
&= \lim_{n\to\infty} \left[\frac{216}{n^3}\sum_{i=1}^{n} i^2 - \frac{252}{n^2}\sum_{i=1}^{n} i + \frac{96}{n}\sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \left[\frac{216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{252}{n^2} \cdot \frac{n(n+1)}{2} + \frac{96}{n} \cdot n\right] \\
&= \lim_{n\to\infty} \left[\frac{36(n+1)(2n+1)}{n^2} - \frac{126(n+1)}{n} + 96\right] \\
&= 72 - 126 + 96 \\
&= 42.
\end{aligned}$$

[5] (b) We form a regular partition where

$$\Delta x = \frac{\frac{1}{3} - 0}{n} = \frac{1}{3n}$$

and choose the sample point

$$x_i^* = 0 + i \cdot \frac{1}{3n} = \frac{i}{3n}.$$

Thus

$$f(x_i^*) = \left(\frac{i}{3n}\right)^2 \left(\frac{i}{3n} + 2\right) = \frac{i^3}{27n^3} + \frac{2i^2}{9n^2}.$$

Now we have

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{i^3}{27n^3} + \frac{2i^2}{9n^2}\right] \cdot \frac{1}{3n} \\
&= \lim_{n\to\infty} \left[\frac{1}{81n^4} \sum_{i=1}^{n} i^3 + \frac{2}{27n^3} \sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{81n^4} \cdot \frac{n^2(n+1)^2}{4} + \frac{2}{27n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left[\frac{(n+1)^2}{324n^2} + \frac{(n+1)(2n+1)}{81n^2}\right] \\
&= \frac{1}{324} + \frac{2}{81} \\
&= \frac{1}{36}.
\end{aligned}$$

[5] 2. Since we are using an irregular partition, the width of the ith subinterval will be

$$\Delta x_i = x_i - x_{i-1} = \frac{16i^2}{n^2} - \frac{16(i-1)^2}{n^2} = \frac{32i}{n^2} - \frac{16}{n^2}.$$

The height of the ith subinterval will be

$$f(x_i^*) = \sqrt{\frac{16i^2}{n^2}} = \frac{4i}{n}.$$

Now we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4i}{n} \cdot \left(\frac{32i}{n^2} - \frac{16}{n^2}\right) \\ &= \lim_{n\to\infty} \left[\frac{128}{n^3} \sum_{i=1}^{n} i^2 - \frac{64}{n^3} \sum_{i=1}^{n} i\right] \\ &= \lim_{n\to\infty} \left[\frac{128}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{64}{n^3} \cdot \frac{n(n+1)}{2}\right] \\ &= \lim_{n\to\infty} \left[\frac{64(n+1)(2n+1)}{3n^2} - \frac{32(n+1)}{n^2}\right] \\ &= \frac{128}{3} - 0 \\ &= \frac{128}{3}. \end{aligned}$$

[5] 3. We use a regular partition of the interval $[-2, 1]$ into n subintervals of width

$$\Delta x = \frac{1-(-2)}{n} = \frac{3}{n}.$$

We choose as our sample point the right endpoint of the ith subinterval,

$$x_i^* = -2 + \frac{3i}{n} = \frac{3i}{n} - 2$$

so that

$$f(x_i^*) = \left(\frac{3i}{n} - 2 - 1\right)^2 = \left(\frac{3i}{n} - 3\right)^2 = \frac{9i^2}{n^2} - \frac{18i}{n} + 9.$$

Then

$$\begin{aligned} \int_{-2}^{1} (x-1)^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{18i}{n} + 9\right) \cdot \frac{3}{n} \\ &= \lim_{n\to\infty} \left[\frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{54}{n^2} \sum_{i=1}^{n} i + \frac{27}{n} \sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty} \left[\frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{54}{n^2} \cdot \frac{n(n+1)}{2} + \frac{27}{n} \cdot n\right] \\ &= 9 - 27 + 27 \\ &= 9. \end{aligned}$$