MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.5 | **Math 1001 Worksheet** | WINTER 2024

SOLUTIONS

1. (a) Observe that $f(x) < 0$ when $0 \le x < 1$. Hence it is not a non-negative function for all x, and therefore it is not a probability density function.

   (b) We can see that $f(x) \ge 0$ for all x, so we evaluate

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_0^8 \frac{1}{(x+4)^3}\,dx \\ &= \left[-\frac{1}{2(x+4)^2}\right]_0^8 \\ &= -\frac{1}{288} + \frac{1}{32} \\ &= \frac{1}{36}.\end{aligned}$$

   Since $\int_{-\infty}^{\infty} f(x)\,dx \neq 1$, this is not a probability density function.

   (c) Again we can see that $f(x) \ge 0$ for all x, so we evaluate

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^8 \frac{18x}{(x+4)^3}\,dx.$$

   We let $u = x+4$ so $du = dx$ and $x = u-4$. When $x = 0$, $u = 4$. When $x = 8$, $u = 12$. The integral becomes

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= 18\int_4^{12} \frac{u-4}{u^3}\,du \\ &= 18\int_4^{12} \left(\frac{1}{u^2} - \frac{4}{u^3}\right)du \\ &= 18\left[-\frac{1}{u} + \frac{2}{u^2}\right]_4^{12} \\ &= 18\left[-\frac{1}{12} + \frac{1}{72} + \frac{1}{4} - \frac{1}{8}\right] \\ &= 1.\end{aligned}$$

   Hence this is a probability density function.

2. (a) Note that $f(x) \geq 0$ for all x as long as $k \geq 0$, and that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{\infty} \frac{k}{x^2+1}\,dx \\
&= \int_{-\infty}^{0} \frac{k}{x^2+1}\,dx + \int_{0}^{\infty} \frac{k}{x^2+1}\,dx \\
&= k \lim_{T\to-\infty} \int_{T}^{0} \frac{1}{x^2+1}\,dx + k \lim_{S\to\infty} \int_{0}^{S} \frac{1}{x^2+1}\,dx \\
&= k \lim_{T\to-\infty} \Big[\arctan(x)\Big]_{T}^{0} + k \lim_{S\to\infty} \Big[\arctan(x)\Big]_{0}^{S} \\
&= k \lim_{T\to-\infty} [0 - \arctan(T)] + k \lim_{S\to\infty} [\arctan(S) - 0] \\
&= k \cdot \frac{\pi}{2} + k \cdot \frac{\pi}{2} \\
&= k\pi.
\end{aligned}$$

Thus we must have $k\pi = 1$ and so $k = \frac{1}{\pi}$.

(b) We have

$$\begin{aligned}
P(-\sqrt{3} \leq X \leq \sqrt{3}) &= \int_{-\sqrt{3}}^{\sqrt{3}} f(x)\,dx \\
&= \frac{1}{\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{x^2+1}\,dx \\
&= \frac{1}{\pi} \Big[\arctan(x)\Big]_{-\sqrt{3}}^{\sqrt{3}} \\
&= \frac{1}{\pi} \left[\arctan\left(\sqrt{3}\right) - \arctan\left(-\sqrt{3}\right)\right] \\
&= \frac{1}{\pi} \left[\frac{\pi}{3} + \frac{\pi}{3}\right] \\
&= \frac{2}{3}.
\end{aligned}$$

3. (a) Note that $f(x) \geq 0$ for all x as long as $k \geq 0$, and that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= k \int_{0}^{\infty} xe^{-2x}\,dx \\
&= k \lim_{T\to\infty} \int_{0}^{T} xe^{-2x}\,dx.
\end{aligned}$$

We use integration by parts with $w = x$ so $dw = dx$, and $dv = e^{-2x}\,dx$ so $v = -\frac{1}{2}e^{-2x}$.

Then

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= k \lim_{T\to\infty} \left(\left[-\frac{1}{2} x e^{-2x} \right]_0^T + \frac{1}{2} \int_0^T e^{-2x}\,dx \right) \\
&= k \lim_{T\to\infty} \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^T \\
&= k \lim_{T\to\infty} \left[-\frac{1}{2} T e^{-2T} - \frac{1}{4} e^{-2T} + 0 + \frac{1}{4} \right] \\
&= \frac{1}{4} k - \frac{1}{2} k \lim_{T\to\infty} \frac{T}{e^{2T}} \\
&\overset{\text{H}}{=} \frac{1}{4} k - \frac{1}{2} k \lim_{T\to\infty} \frac{1}{2e^{2T}} \\
&= \frac{1}{4} k - 0 \\
&= \frac{1}{4} k.
\end{aligned}
$$

Hence we must have $\frac{1}{4}k = 1$ and so $k = 4.$

(b) We have

$$
\begin{aligned}
P(0 \le X \le 2) &= \int_0^1 f(x)\,dx \\
&= 4 \int_0^2 x e^{-2x}\,dx \\
&= 4 \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^2 \\
&= 4 \left[-e^{-4} - \frac{1}{4} e^{-4} + \frac{1}{4} \right] \\
&= 1 - 5e^{-4} \\
&\approx 0.9.
\end{aligned}
$$

4. (a) Using the same substitution as in Question 1(c), this probability is given by

$$\begin{aligned}
P(3 \le X \le 5) &= \int_3^5 f(x)\,dx \\
&= 18 \int_3^5 \frac{x}{(x+4)^3}\,dx \\
&= 18 \int_7^9 \left(\frac{1}{u^2} - \frac{4}{u^3}\right) du \\
&= 18 \left[-\frac{1}{u} + \frac{2}{u^2}\right]_7^9 \\
&= 18 \left[-\frac{1}{9} + \frac{2}{81} + \frac{1}{7} - \frac{2}{49}\right] \\
&= \frac{124}{441}.
\end{aligned}$$

Hence the likelihood is about 28.1% that the total viewing time will be between 3 minutes and 5 minutes.

(b) Using the same substitution as in Question 1(c), this probability is given by

$$\begin{aligned}
P(0 \le X \le 1) &= \int_0^1 f(x)\,dx \\
&= 18 \int_0^1 \frac{x}{(x+4)^3}
\end{aligned}$$

, dx

$$\begin{aligned}
&= 18 \int_4^5 \left(\frac{1}{u^2} - \frac{4}{u^3}\right) du \\
&= 18 \left[-\frac{1}{u} + \frac{2}{u^2}\right]_4^5 \\
&= 18 \left[-\frac{1}{5} + \frac{2}{25} + \frac{1}{4} - \frac{1}{8}\right] \\
&= \frac{9}{100}.
\end{aligned}$$

Hence there's a 9% likelihood that the total viewing time will be less than 1 minute.

(c) Using the same substitution as in Question 1(c), this probability is given by

$$\begin{aligned} P(6 \leq X \leq 8) &= \int_6^8 f(x)\,dx \\ &= 18 \int_6^8 \frac{x}{(x+4)^3} \end{aligned}$$

, dx

$$\begin{aligned} &= 18 \int_{10}^{12} \left(\frac{1}{u^2} - \frac{4}{u^3} \right) du \\ &= 18 \left[-\frac{1}{u} + \frac{2}{u^2} \right]_{10}^{12} \\ &= 18 \left[-\frac{1}{12} + \frac{1}{72} + \frac{1}{10} - \frac{1}{50} \right] \\ &= \frac{19}{100}. \end{aligned}$$

Hence there's a 19% likelihood that the total viewing time will be greater than 6 minutes.

(d) We have

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_0^8 x \cdot \frac{18x}{(x+4)^3}\,dx \\ &= 18 \int_0^8 \frac{x^2}{(x+4)^3}\,dx. \end{aligned}$$

Let $u = x + 4$ so $du = dx$ and $x^2 = (u-4)^2$. When $x = 0$, $u = 4$. When $x = 8$, $u = 12$.

The integral becomes

$$\begin{aligned}
\mu &= 18\int_4^{12} \frac{(u-4)^2}{u^3}\,du \\
&= 18\int_4^{12} \frac{u^2-8u+16}{u^3}\,du \\
&= 18\int_4^{12} \left(\frac{1}{u}-\frac{8}{u^2}+\frac{16}{u^3}\right)\,du \\
&= 18\left[\ln|u|+\frac{8}{u}-\frac{8}{u^2}\right]_4^{12} \\
&= 18\left[\ln(12)+\frac{2}{3}-\frac{1}{18}-\ln(4)-2+\frac{1}{2}\right] \\
&= 18\ln(12)-18\ln(4)-16 \\
&\approx 3.8.
\end{aligned}$$

5. (a) Clearly, $f(x) \geq 0$ for all x, and furthermore

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_0^5 \frac{1}{5}\,dx \\
&= \frac{1}{5}\Big[x\Big]_0^5 \\
&= \frac{1}{5}\cdot 5 \\
&= 1.
\end{aligned}$$

Hence $f(x)$ is a probability density function.
Next,

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} xf(x)\,dx \\
&= \int_0^5 \frac{1}{5}x\,dx \\
&= \frac{1}{5}\left[\frac{1}{2}x^2\right]_0^5 \\
&= \frac{1}{5}\left[\frac{25}{2}-0\right] \\
&= \frac{5}{2}.
\end{aligned}$$

Note that this makes sense, since it's the midpoint of the interval $[0, 5]$.

(b) We want a density function such that, for some constant k,

$$f(x) = \begin{cases} k, & \text{for } 0 \le x < 4 \\ 2k, & \text{for } 4 \le x \le 5 \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $f(x) \ge 0$ for all x as long $k \ge 0$. Then we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_0^4 k\,dx + \int_4^5 2k\,dx \\ &= k\Big[x\Big]_0^4 + 2k\Big[x\Big]_4^5 \\ &= k(4-0) + 2k(5-4) \\ &= 6k, \end{aligned}$$

so we need $6k = 1$ therefore $k = \frac{1}{6}$. Hence a suitable probability density function is given by

$$\boxed{f(x) = \begin{cases} \frac{1}{6}, & \text{for } 0 \le x < 4 \\ \frac{1}{3}, & \text{for } 4 \le x \le 5 \\ 0, & \text{otherwise.} \end{cases}}$$

Finally,

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \frac{1}{6}\int_0^4 x\,dx + \frac{1}{3}\int_4^5 x\,dx \\ &= \frac{1}{6}\left[\frac{1}{2}x^2\right]_0^4 + \frac{1}{3}\left[\frac{1}{2}x^2\right]_4^5 \\ &= \frac{1}{6}[8-0] + \frac{1}{3}\left[\frac{25}{2} - 8\right] \\ &\boxed{= \frac{17}{6}.} \end{aligned}$$

In other words, the effect of Ruby's error is to shift the mean value of the probability distribution from 2.5 to $2.8\overline{3}$.