MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.4 **Math 1001 Worksheet** WINTER 2024

SOLUTIONS

1. (a) Note that $x(t)$ is negatively impacted by the interactions between the two populations, as indicated by the term $-2xy$ in the first differential equation, while $y(t)$ is positively impacted, as indicated by the term $+3xy$ in the second differential equation. Hence we can conclude that $x(t)$ is the prey and $y(t)$ is the predator.

   (b) If $y = 0$ then $x(t)$ is described by the differential equation
   $$\frac{dx}{dt} = 5x,$$
   which indicates exponential growth. Hence no other factor inhibits the growth of the prey population.

   (c) First we set $5x - 2xy = 0$ so $x(5 - 2y) = 0$ and either $\overline{x} = 0$ or $\overline{y} = \frac{5}{2}$. Now we consider $-y + 3xy = 0$. If $\overline{x} = 0$ then this becomes $-y = 0$ so $\overline{y} = 0$, If $\overline{y} = \frac{5}{2}$ then it becomes
   $$-\frac{5}{2} + 3x \cdot \frac{5}{2} = 0 \quad \Longrightarrow \quad \overline{x} = \frac{1}{3}.$$
   Hence the equilibrium points are $(0, 0)$ which indicates the extinction of both populations, and $(\frac{1}{3}, \frac{5}{2})$ which indicates that a prey population of $x(t) = \frac{1}{3}$ perfectly supports a predator population of $y(t) = \frac{5}{2}$.

2. (a) Note that $x(t)$ is positively impacted by the interactions between the two populations, as indicated by the term $+8xy$ in the first differential equation, while $y(t)$ is negatively impacted, as indicated by the term $-xy$ in the second differential equation. Hence we can conclude that $x(t)$ is the predator and $y(t)$ is the prey.

   (b) If $x = 0$ then $y(t)$ is described by the differential equation
   $$\frac{dy}{dt} = 3y - 6y^2 = 3y(1 - 2y),$$
   which is the form of the logistic model. Hence the prey population is inhibited by environmental factors such as food availability, in addition to the actions of the predator population.

   (c) First we set $-4x + 8xy = 0$ so $x(-4 + 8y) = 0$ and either $\overline{x} = 0$ or $\overline{y} = \frac{1}{2}$. Now we consider $3y - 6y^2 - xy = 0$. If $\overline{x} = 0$ then this becomes
   $$3y - 6y^2 = 0 \quad \Longrightarrow \quad 3y(1 - 2y) = 0$$

so either $\overline{y} = 0$ or $\overline{y} = \frac{1}{2}$. If $\overline{y} = \frac{1}{2}$ then it becomes

$$3 \cdot \frac{1}{2} - 6 \cdot \frac{1}{4} - x \cdot \frac{1}{2} = 0 \quad \Longrightarrow \quad \frac{1}{2}x = 0$$

so $\overline{x} = 0$, as we've already determined. Hence the equilibrium points are $(0, 0)$ which indicates the extinction of both populations, and $(0, \frac{1}{2})$ which indicates the extinction of the predator population and the survival of the prey population.

3. (a) Since their interactions positively impact both populations, as indicated by the term $+8xy$ in the first differential equation and the term $+xy$ in the second differential equation, this model describes co-operation.

   (b) First we set $-4x + 4xy = 0$ so $x(-4 + 4y) = 0$ and either $\overline{x} = 0$ or $\overline{y} = 1$. Now we consider $3y - 6y^2 + xy = 0$. If $\overline{x} = 0$ then this becomes

$$3y - 6y^2 = 0 \quad \Longrightarrow \quad 3y(1 - 2y) = 0$$

so either $\overline{y} = 0$ or $\overline{y} = \frac{1}{2}$. If $\overline{y} = 1$ then it becomes

$$3 \cdot 1 - 6 \cdot 1 + x \cdot 1 = 0$$

so $\overline{x} = 3$. Hence the equilibrium points are $(0, 0)$ which indicates the extinction of both populations, $(0, \frac{1}{2})$ which indicates the extinction of $x(t)$ and the survival of $y(t)$, and $(3, 1)$ which indicates the survival of both populations.

4. (a) Since their interactions negatively impact both populations, as indicated by the term $-5xy$ in the first differential equation and the term $-4xy$ in the second differential equation, this model describes competition.

   (b) First we set $6x - 2x^2 - 5xy = 0$ so $x(6 - 2x - 5y) = 0$ and either $\overline{x} = 0$ or $\overline{x} = 3 - \frac{5}{2}y$. Now we consider $8y - 5y^2 - 4xy = 0$. If $\overline{x} = 0$ then this becomes

$$8y - 5y^2 = 0 \quad \Longrightarrow \quad y(8 - 5y) = 0$$

so either $\overline{y} = 0$ or $\overline{y} = \frac{8}{5}$. If $\overline{x} = 3 - \frac{5}{2}y$ then it becomes

$$8y - 5y^2 - 4\left(3 - \frac{5}{2}y\right)y = 0$$

$$8y - 5y^2 - 12y + 10y^2 = 0$$

$$5y^2 - 4y = 0$$

$$y(5y - 4) = 0$$

so either $\overline{y} = 0$, in which case $\overline{x} = 3$, or $\overline{y} = \frac{4}{5}$, in which case $\overline{x} = 1$. Hence we have four equilibrium points. First, $(0, 0)$ indicates the extinction of both populations. Second, $(0, \frac{8}{5})$ indicates the extinction of $x(t)$ and the survival of $y(t)$. Third, $(3, 0)$ indicates the extinction of $y(t)$ and the survival of $x(t)$. Finally, $(1, \frac{4}{5})$ indicates the survival of both populations.