SOLUTIONS

1. (a) The equation becomes

$$\begin{aligned}\frac{t}{y^2+1}\frac{dy}{dt} &= -\frac{t^2}{e^t}\\ \frac{1}{y^2+1}\,dy &= -te^{-t}\,dt\\ \int \frac{1}{y^2+1}\,dy &= -\int te^{-t}\,dt\\ \arctan(y) &= -\int te^{-t}\,dt.\end{aligned}$$

The integral on the righthand side can be evaluated using integration by parts, with $w = t$ so $dw = dt$ and $dv = e^{-t}\,dt$ so $v = -e^{-t}$. Thus

$$\int te^{-t}\,dt = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C$$

and so the solution is

$$\arctan(y) = -(-te^{-t} - e^{-t}) + C \quad \Longrightarrow \quad \boxed{y = \tan(te^{-t} + e^{-t} + C).}$$

(b) We can rewrite the given equation as

$$\begin{aligned}\frac{5}{(y+2)(y-3)}\,dy &= \frac{1}{\cos(t)\csc(t)}\,dt\\ \frac{5}{(y+2)(y-3)}\,dy &= \frac{1}{\cos(t)\cdot\frac{1}{\sin(t)}}\,dt\\ \frac{5}{(y+2)(y-3)}\,dy &= \frac{\sin(t)}{\cos(t)}\,dt\\ \frac{5}{(y+2)(y-3)}\,dy &= \tan(t)\,dt\\ \int\frac{5}{(y+2)(y-3)}\,dy &= \int\tan(t)\,dt\\ \int\frac{5}{(y+2)(y-3)}\,dy &= -\ln|\cos(t)| + C.\end{aligned}$$

The lefthand side requires the method of partial fractions. We know that

$$\frac{5}{(y+2)(y-3)} = \frac{A}{y+2} + \frac{B}{y-3}$$
$$5 = A(y-3) + B(y+2).$$

When $y = -2$, we have $5 = -5A$ so $A = -1$. When $y = 3$, we have $5 = 5B$ so $B = 1$. Thus

$$\int \frac{5}{(y+2)(y-3)}\,dy = \int \left(\frac{-1}{y+2} + \frac{1}{y-3}\right) dy = \ln|y-3| - \ln|y+2| + C.$$

Hence the solution to the differential equation is given by the implicit form

$$\ln|y-3| - \ln|y+2| = -\ln|\cos(t)| + C.$$

This is sufficient, but note that we can simplify this to an extent. If we replace C with $\ln|C|$ and apply the properties of logarithms, we have

$$\ln|y-3| - \ln|y+2| = -\ln|\cos(t)| + \ln|C|$$
$$\ln\left|\frac{y-3}{y+2}\right| = \ln\left|\frac{C}{\cos(t)}\right|$$
$$\frac{y-3}{y+2} = \frac{C}{\cos(t)},$$

which is a bit simpler than our original answer.

2. The differential equation is separable, so it can be rewritten as

$$t\frac{dy}{dt} = \sqrt{4-y^2}$$
$$\frac{1}{\sqrt{4-y^2}}\,dy = \frac{1}{t}\,dt$$
$$\int \frac{1}{\sqrt{4-y^2}}\,dy = \int \frac{1}{t}\,dt$$
$$\arcsin\left(\frac{y}{2}\right) = \ln|t| + C.$$

Since $y(1) = 2$, we have

$$\arcsin(1) = \ln(1) + C$$
$$\frac{\pi}{2} = C.$$

Thus the particular solution is

$$\arcsin\left(\frac{y}{2}\right) = \ln|t| + \frac{\pi}{2} \quad \text{or} \quad y = 2\sin\left(\ln|t| + \frac{\pi}{2}\right).$$

3. (a) Let $y(t)$ be the amount of Einsteinium-254 left, in milligrams, after t days, so $y = Ce^{kt}$. Then we first know that $y(0) = C = 3$. Since half the sample is left after 270 days, this implies

$$y(270) = 3e^{270k} = 1.5 \quad \Longrightarrow \quad k = -\frac{1}{270}\ln(2) \quad \Longrightarrow \quad y = 3e^{-\frac{t}{270}\ln(2)}.$$

Thus, after 30 days, there will be

$$y(30) = 3e^{-\frac{30}{270}\ln(2)} = 3e^{-\frac{1}{9}\ln(2)} = \frac{3}{2^{\frac{1}{9}}} \approx 2.78.$$

There will be about 2.78 mg left after 30 days.

(b) We want to know when $y = 0.5$ so, from part (a), we set

$$3e^{-\frac{t}{270}\ln(2)} = 0.5 \quad \Longrightarrow \quad t = \frac{270\ln(6)}{\ln(2)} \approx 698.$$

So it takes about 698 days for the sample to be reduced to 0.5 mg.

4. Let $y(t)$ be the number of parakeets on the island after t years, so $y = Ce^{kt}$. Then we know that

$$y(2) = Ce^{2k} = 50 \quad \text{and} \quad y(5) = Ce^{5k} = 150.$$

Dividing the second by the first gives

$$\frac{Ce^{5k}}{Ce^{2k}} = \frac{150}{50} \quad \Longrightarrow \quad e^{3k} = 3 \quad \Longrightarrow \quad k = \frac{1}{3}\ln(3).$$

Hence using $y(2) = 50$ we have

$$50 = Ce^{\frac{2}{3}\ln(3)} \quad \Longrightarrow \quad C = 50e^{-\frac{2}{3}\ln(3)} \approx 24.$$

So there were about 24 parakeets originally on the island.

5. Let $y(t)$ be the number of healthy individuals after t days, so $y = Ce^{kt}$. If there are C people in the city, after 10 days 10% (that is, $\frac{1}{10}$) of them have contracted the flu, so only $\frac{9}{10}C$ people are healthy. Hence

$$y(10) = Ce^{10k} = \frac{9}{10}C \quad \Longrightarrow \quad e^{10k} = \frac{9}{10} \quad \Longrightarrow \quad k = \frac{1}{10}\ln\left(\frac{9}{10}\right).$$

We want to know when 40% of the population is infected, so 60% or $\frac{3}{5}C$ people remain healthy, implying

$$y = Ce^{\frac{t}{10}\ln\left(\frac{9}{10}\right)} = \frac{3}{5}C \quad \Longrightarrow \quad e^{\frac{t}{10}\ln\left(\frac{9}{10}\right)} = \frac{3}{5} \quad \Longrightarrow \quad t = 10\frac{\ln\left(\frac{3}{5}\right)}{\ln\left(\frac{9}{10}\right)} \approx 48.5.$$

So it will take about 48.5 days for 40% of the people to contract the flu.

6. (a) Newton's Law of Cooling can be represented by the differential equation

$$\frac{dy}{dt} = k(y - T),$$

which is separable. It can be rewritten

$$\begin{aligned} \frac{1}{y-T}\,dy &= k\,dt \\ \int \frac{1}{y-T}\,dy &= \int k\,dt \\ \ln(y-T) &= kt + C \\ y - T &= Ce^{kt} \\ y &= Ce^{kt} + T. \end{aligned}$$

In this case, $y(0) = C + T$ so $C = T - y(0) = T - y_0$. Thus the particular solution is given by

$$\boxed{y = (T - y_0)e^{kt} + T.}$$

(b) Here $y_0 = 37$ and $T = -8$, so the temperature of the body is given by

$$y = 45e^{kt} - 8.$$

Furthermore,

$$y(30) = 45e^{30k} - 8 = 25 \quad \Longrightarrow \quad k = \frac{1}{30}\ln\left(\frac{33}{45}\right).$$

Hence

$$y(45) = 45e^{\frac{45}{30}\ln\left(\frac{33}{45}\right)} - 8 \approx 20.3.$$

So the temperature of the body when the medical examiner arrives is about $\boxed{20.3°\text{C}.}$