MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 | **MATHEMATICS 1001-004** | WINTER 2024

SOLUTIONS

[7] 1. (a) We use a regular partition with subintervals of width

$$\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}.$$

We choose the sample point

$$x_i = -1 + i\Delta x = \frac{2i}{n} - 1.$$

Thus

$$\begin{aligned} f(x_i) &= 6\left(\frac{2i}{n} - 1\right)^2 - \left(\frac{2i}{n} - 1\right) + 2 \\ &= 6\left(\frac{4i^2}{n^2} - \frac{4i}{n} + 1\right) - \frac{2i}{n} + 1 + 2 \\ &= \frac{24i^2}{n^2} - \frac{26i}{n} + 9. \end{aligned}$$

Now we can write

$$\begin{aligned} \int_{-1}^{1} (6x^2 - x + 2)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{24i^2}{n^2} - \frac{26i}{n} + 9\right) \cdot \frac{2}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{48i^2}{n^3} - \frac{52i}{n^2} + \frac{18}{n}\right] \\ &= \lim_{n\to\infty} \left[\frac{48}{n^3} \sum_{i=1}^{n} i^2 - \frac{52}{n^2} \sum_{i=1}^{n} i + \frac{18}{n} \sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty} \left[\frac{48}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{52}{n^2} \cdot \frac{n(n+1)}{2} + \frac{18}{n} \cdot n\right] \\ &= 16 - 26 + 18 \\ &= 8. \end{aligned}$$

[3] (b) We have

$$\int_{-1}^{1}(6x^2 - x + 2)\,dx = \left[6 \cdot \frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-1}^{1}$$

$$= \left[2x^3 - \frac{1}{2}x^2 + 2x\right]_{-1}^{1}$$

$$= \left[2 - \frac{1}{2} + 2\right] - \left[-2 - \frac{1}{2} - 2\right]$$

$$= 8.$$

[5] 2. (a) The integrand is an improper rational function, so we can use long division, or take advantage of the similarity between the numerator and the denominator to write

$$\int_0^3 \frac{x^2+4}{x^2+9}\,dx = \int_0^3 \frac{x^2+9-5}{x^2+9}\,dx$$

$$= \int_0^3 \left(1 - \frac{5}{x^2+9}\right)dx$$

$$= \left[x - 5 \cdot \frac{1}{3}\arctan\left(\frac{x}{3}\right)\right]_0^3$$

$$= \left[3 - \frac{5}{3}\arctan(1)\right] - \left[0 - \frac{5}{3}\arctan(0)\right]$$

$$= 3 - \frac{5}{3} \cdot \frac{\pi}{4} + \frac{5}{3} \cdot 0$$

$$= 3 - \frac{5\pi}{12}.$$

[5] (b) Let $u = x^2 + 9$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. When $x = 0$, $u = 0^2 + 9 = 9$. When $x = 4$, $u = 4^2 + 9 = 25$. Thus the integral becomes

$$\int_0^4 \frac{3x}{\sqrt{x^2+9}}\,dx = \frac{3}{2}\int_9^{25} u^{-\frac{1}{2}}\,du$$

$$= \frac{3}{2}\left[2\sqrt{u}\right]_9^{25}$$

$$= \frac{3}{2}\left[2\sqrt{25} - 2\sqrt{9}\right]$$

$$= 6.$$

[5] (c) Using the Additive Interval Property, we have

$$\begin{aligned}
\int_{-\frac{1}{2}}^{2} f(x)\,dx &= \int_{-\frac{1}{2}}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx \\
&= \int_{-\frac{1}{2}}^{0} e^{2x}\,dx + \int_{0}^{1} \cos(\pi x)\,dx + \int_{1}^{2} \left(-\frac{1}{x^2}\right)\,dx \\
&= \left[\frac{e^{2x}}{2}\right]_{-\frac{1}{2}}^{0} + \left[\frac{\sin(\pi x)}{\pi}\right]_{0}^{1} + \left[\frac{1}{x}\right]_{1}^{2} \\
&= \left[\frac{e^0}{2} - \frac{e^{-2}}{2}\right] + \left[\frac{\sin(\pi)}{\pi} - \frac{\sin(0)}{\pi}\right] + \left[\frac{1}{2} - 1\right] \\
&= \frac{1}{2} - \frac{1}{2e} + 0 - 0 + \frac{1}{2} - 1 \\
&= -\frac{1}{2e}.
\end{aligned}$$

[5] 3. We can write

$$g(x) = \int_{x^2}^{0} t^2\cos(t^4)\,dt + \int_{0}^{x} t^2\cos(t^4)\,dt = -\int_{0}^{x^2} t^2\cos(t^4)\,dt + \int_{0}^{x} t^2\cos(t^4)\,dt.$$

Then, by the First Fundamental Theorem of Calculus and the Chain Rule, we have

$$\begin{aligned}
g'(x) &= -(x^2)^2\cos((x^2)^4)\cdot[x^2]' + x^2\cos(x^4) \\
&= -x^4\cos(x^8)\cdot 2x + x^2\cos(x^4) \\
&= x^2\cos(x^4) - 2x^5\cos(x^8).
\end{aligned}$$

[10] 4. (a) The sketch of R is given in Figure 1. Note that $y = 2 - \frac{1}{2}x$ and $y = \sqrt{x-1}$ intersect when

$$\begin{aligned}
2 - \frac{1}{2}x &= \sqrt{x-1} \\
\left(2 - \frac{1}{2}x\right)^2 &= x - 1 \\
4 - 2x + \frac{1}{4}x^2 &= x - 1 \\
x^2 - 12x + 20 &= 0 \\
(x-10)(x-2) &= 0
\end{aligned}$$

and so $x = 10$ or $x = 2$. However, substituting these values back into the original equation confirms that $x = 10$ is not a solution of the equation. Hence only the point $(2, 1)$ is an intersection point.

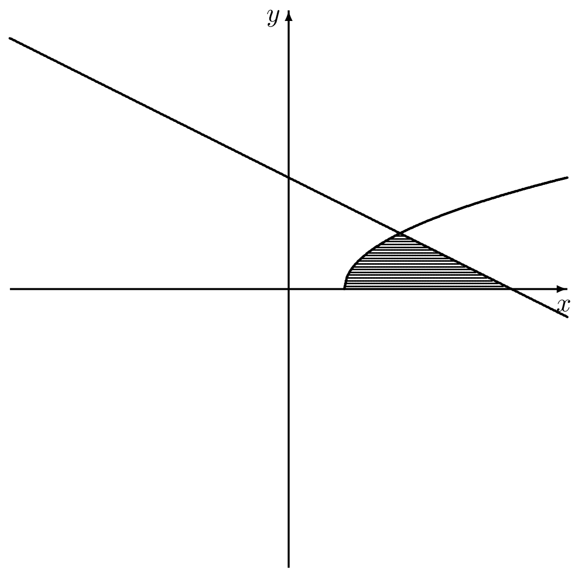

Figure 1: Question 4(a)

(b) From the graph, we can see that the region is always bounded below by the x-axis (that is, the line $y = 0$). On the interval $[1, 2]$ the top boundary curve is $y = \sqrt{x-1}$. On the interval $[2, 4]$ the top boundary curve is $y = 2 - \frac{1}{2}x$. Thus

$$A = \int_1^2 \left[\sqrt{x-1} - 0\right] dx + \int_2^4 \left[\left(2 - \frac{1}{2}x\right) - 0\right] dx$$

$$= \int_1^2 \sqrt{x-1}\, dx + \int_2^4 \left(2 - \frac{1}{2}x\right) dx.$$

(c) The function $y = \sqrt{x-1}$ can be written $x = y^2 + 1$ (with $y \geq 0$), while $y = 2 - \frac{1}{2}x$ becomes $x = 4 - 2y$. The graph shows that $x = 4 - 2y$ is always the rightmost boundary curve, while $x = y^2 + 1$ is always the leftmost boundary curve. So then

$$A = \int_0^1 [(4 - 2y) - (y^2 + 1)]\, dy$$

$$= \int_0^1 (3 - 2y - y^2)\, dy.$$