

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2

MATHEMATICS 1001-002

WINTER 2024

SOLUTIONS

[7] 1. (a) We use a regular partition with subintervals of width

$$\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}.$$

We choose the sample point

$$x_i = -1 + i\Delta x = \frac{2i}{n} - 1.$$

Thus

$$\begin{aligned} f(x_i) &= 6 \left(\frac{2i}{n} - 1 \right)^2 - \left(\frac{2i}{n} - 1 \right) + 2 \\ &= 6 \left(\frac{4i^2}{n^2} - \frac{4i}{n} + 1 \right) - \frac{2i}{n} + 1 + 2 \\ &= \frac{24i^2}{n^2} - \frac{26i}{n} + 9. \end{aligned}$$

Now we can write

$$\begin{aligned} \int_{-1}^1 (6x^2 - x + 2) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{24i^2}{n^2} - \frac{26i}{n} + 9 \right) \cdot \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{48i^2}{n^3} - \frac{52i}{n^2} + \frac{18}{n} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{48}{n^3} \sum_{i=1}^n i^2 - \frac{52}{n^2} \sum_{i=1}^n i + \frac{18}{n} \sum_{i=1}^n 1 \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{48}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{52}{n^2} \cdot \frac{n(n+1)}{2} + \frac{18}{n} \cdot n \right] \\ &= 16 - 26 + 18 \\ &= 8. \end{aligned}$$

[3] (b) We have

$$\begin{aligned}\int_{-1}^1 (6x^2 - x + 2) dx &= \left[6 \cdot \frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-1}^1 \\ &= \left[2x^3 - \frac{1}{2}x^2 + 2x \right]_{-1}^1 \\ &= \left[2 - \frac{1}{2} + 2 \right] - \left[-2 - \frac{1}{2} - 2 \right] \\ &= 8.\end{aligned}$$

[5] 2. (a) The integrand is an improper rational function, so we can use long division, or take advantage of the similarity between the numerator and the denominator to write

$$\begin{aligned}\int_0^3 \frac{x^2 + 4}{x^2 + 9} dx &= \int_0^3 \frac{x^2 + 9 - 5}{x^2 + 9} dx \\ &= \int_0^3 \left(1 - \frac{5}{x^2 + 9} \right) dx \\ &= \left[x - 5 \cdot \frac{1}{3} \arctan\left(\frac{x}{3}\right) \right]_0^3 \\ &= \left[3 - \frac{5}{3} \arctan(1) \right] - \left[0 - \frac{5}{3} \arctan(0) \right] \\ &= 3 - \frac{5}{3} \cdot \frac{\pi}{4} + \frac{5}{3} \cdot 0 \\ &= 3 - \frac{5\pi}{12}.\end{aligned}$$

[5] (b) Let $u = x^2 + 9$ so $du = 2x dx$ and $\frac{1}{2} du = x dx$. When $x = 0$, $u = 0^2 + 9 = 9$. When $x = 4$, $u = 4^2 + 9 = 25$. Thus the integral becomes

$$\begin{aligned}\int_0^4 \frac{3x}{\sqrt{x^2 + 9}} dx &= \frac{3}{2} \int_9^{25} u^{-\frac{1}{2}} du \\ &= \frac{3}{2} \left[2\sqrt{u} \right]_9^{25} \\ &= \frac{3}{2} \left[2\sqrt{25} - 2\sqrt{9} \right] \\ &= 6.\end{aligned}$$

[5] (c) Using the Additive Interval Property, we have

$$\begin{aligned}
 \int_{-\frac{1}{2}}^2 f(x) dx &= \int_{-\frac{1}{2}}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^2 f(x) dx \\
 &= \int_{-\frac{1}{2}}^0 e^{2x} dx + \int_0^1 \cos(\pi x) dx + \int_1^2 \left(-\frac{1}{x^2}\right) dx \\
 &= \left[\frac{e^{2x}}{2}\right]_{-\frac{1}{2}}^0 + \left[\frac{\sin(\pi x)}{\pi}\right]_0^1 + \left[\frac{1}{x}\right]_1^2 \\
 &= \left[\frac{e^0}{2} - \frac{e^{-2}}{2}\right] + \left[\frac{\sin(\pi)}{\pi} - \frac{\sin(0)}{\pi}\right] + \left[\frac{1}{2} - 1\right] \\
 &= \frac{1}{2} - \frac{1}{2e} + 0 - 0 + \frac{1}{2} - 1 \\
 &= -\frac{1}{2e}.
 \end{aligned}$$

[5] 3. We can write

$$g(x) = \int_{x^2}^0 t^2 \cos(t^4) dt + \int_0^x t^2 \cos(t^4) dt = -\int_0^{x^2} t^2 \cos(t^4) dt + \int_0^x t^2 \cos(t^4) dt.$$

Then, by the First Fundamental Theorem of Calculus and the Chain Rule, we have

$$\begin{aligned}
 g'(x) &= -(x^2)^2 \cos((x^2)^4) \cdot [x^2]' + x^2 \cos(x^4) \\
 &= -x^4 \cos(x^8) \cdot 2x + x^2 \cos(x^4) \\
 &= x^2 \cos(x^4) - 2x^5 \cos(x^8).
 \end{aligned}$$

[10] 4. (a) The sketch of R is given in Figure 1. Note that $y = 2 - \frac{1}{2}x$ and $y = \sqrt{x-1}$ intersect when

$$\begin{aligned}
 2 - \frac{1}{2}x &= \sqrt{x-1} \\
 \left(2 - \frac{1}{2}x\right)^2 &= x-1 \\
 4 - 2x + \frac{1}{4}x^2 &= x-1 \\
 x^2 - 12x + 20 &= 0 \\
 (x-10)(x-2) &= 0
 \end{aligned}$$

and so $x = 10$ or $x = 2$. However, substituting these values back into the original equation confirms that $x = 10$ is not a solution of the equation. Hence only the point $(2, 1)$ is an intersection point.

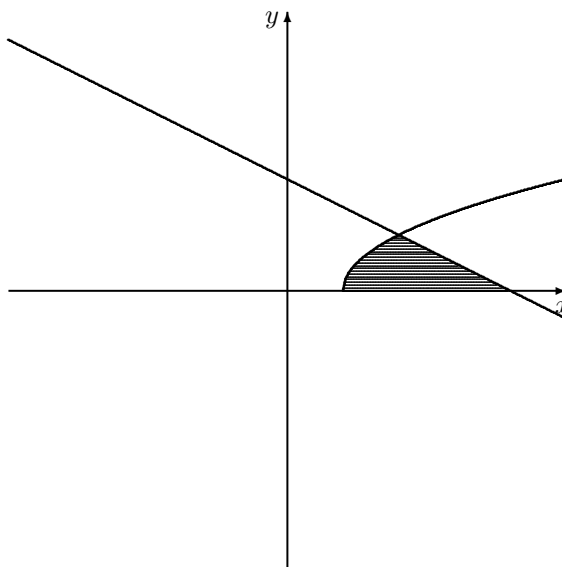


Figure 1: Question 4(a)

- (b) From the graph, we can see that the region is always bounded below by the x -axis (that is, the line $y = 0$). On the interval $[1, 2]$ the top boundary curve is $y = \sqrt{x-1}$. On the interval $[2, 4]$ the top boundary curve is $y = 2 - \frac{1}{2}x$. Thus

$$\begin{aligned}
 A &= \int_1^2 [\sqrt{x-1} - 0] dx + \int_2^4 \left[\left(2 - \frac{1}{2}x\right) - 0 \right] dx \\
 &= \int_1^2 \sqrt{x-1} dx + \int_2^4 \left(2 - \frac{1}{2}x\right) dx.
 \end{aligned}$$

- (c) The function $y = \sqrt{x-1}$ can be written $x = y^2 + 1$ (with $y \geq 0$), while $y = 2 - \frac{1}{2}x$ becomes $x = 4 - 2y$. The graph shows that $x = 4 - 2y$ is always the rightmost boundary curve, while $x = y^2 + 1$ is always the leftmost boundary curve. So then

$$\begin{aligned}
 A &= \int_0^1 [(4 - 2y) - (y^2 + 1)] dy \\
 &= \int_0^1 (3 - 2y - y^2) dy.
 \end{aligned}$$