

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2

MATHEMATICS 1001-001

WINTER 2026

SOLUTIONS

- [7] 1. (a) We use a regular partition with subintervals of width

$$\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}.$$

We choose the sample point

$$x_i^* = x_i = -1 + i\Delta x = \frac{2i}{n} - 1.$$

Thus

$$\begin{aligned} f(x_i^*) &= \left[3 \left(\frac{2i}{n} - 1 \right) - 1 \right]^2 \\ &= \left[\frac{6i}{n} - 4 \right]^2 \\ &= \frac{36i^2}{n^2} - \frac{48i}{n} + 16. \end{aligned}$$

Now we can write

$$\begin{aligned} \int_{-1}^1 (3x - 1)^2 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{36i^2}{n^2} - \frac{48i}{n} + 16 \right) \cdot \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \left[\frac{72}{n^3} \sum_{i=1}^n i^2 - \frac{96}{n^2} \sum_{i=1}^n i + \frac{32}{n} \sum_{i=1}^n 1 \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{72}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{96}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n} \cdot n \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{12(n+1)(2n+1)}{n^2} - \frac{48(n+1)}{n} + 32 \right] \\ &= 24 - 48 + 32 \\ &= 8. \end{aligned}$$

[3] (b) We have

$$\begin{aligned}\int_{-1}^1 (3x - 1)^2 dx &= \int_{-1}^1 (9x^2 - 6x + 1) dx \\&= \left[9 \cdot \frac{x^3}{3} - 6 \cdot \frac{x^2}{2} + x \right]_{-1}^1 \\&= \left[3x^3 - 3x^2 + x \right]_{-1}^1 \\&= (3 - 3 + 1) - (-3 - 3 - 1) \\&= 8.\end{aligned}$$

[6] 2. (a) First we rewrite the integral as

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2(x) \cos^3(x) dx &= \int_0^{\frac{\pi}{2}} \sin^2(x) \cos^2(x) \cdot \cos(x) dx \\&= \int_0^{\frac{\pi}{2}} \sin^2(x) [1 - \sin^2(x)] \cdot \cos(x) dx.\end{aligned}$$

Now we let $u = \sin(x)$ so $du = \cos(x) dx$. When $x = 0$, $u = \sin(0) = 0$. When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$. Thus the integral becomes

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2(x) \cos^3(x) dx &= \int_0^1 u^2(1 - u^2) du \\&= \int_0^1 (u^2 - u^4) du \\&= \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1 \\&= \left(\frac{1}{3} - \frac{1}{5} \right) - (0 - 0) \\&= \frac{2}{15}.\end{aligned}$$

- [5] (b) Using the Additive Interval Property,

$$\begin{aligned}
 \int_0^4 f(x) dx &= \int_0^1 f(x) dx + \int_1^4 f(x) dx \\
 &= \int_0^1 \frac{1}{x^2 + 1} dx + \int_1^4 \frac{1}{2\sqrt{x}} dx \\
 &= \left[\arctan(x) \right]_0^1 + \left[\sqrt{x} \right]_1^4 \\
 &= [\arctan(1) - \arctan(0)] + [\sqrt{4} - \sqrt{1}] \\
 &= \frac{\pi}{4} + 1.
 \end{aligned}$$

- [6] (c) Since the integrand is a proper rational function, we use the method of partial fractions. The partial fraction decomposition has the form

$$\begin{aligned}
 \frac{6x^2 + 9}{x^2(x - 3)} &= \frac{A}{x - 3} + \frac{Bx + C}{x^2} \\
 6x^2 + 9 &= Ax^2 + (Bx + C)(x - 3).
 \end{aligned}$$

When $x = 0$, we have $9 = -3C$ so $C = -3$. When $x = 3$, we have $63 = 9A$ so $A = 7$. Finally we can use, say, $x = 1$ and get

$$15 = A - 2(B + C) \implies 15 = 7 - 2B + 6 \implies 2 = -2B \implies B = -1.$$

Thus

$$\begin{aligned}
 \int \frac{6x^2 + 9}{x^2(x - 3)} dx &= \int \left(\frac{7}{x - 3} + \frac{-x - 3}{x^2} \right) dx \\
 &= \int \left(\frac{7}{x - 3} - \frac{1}{x} - \frac{3}{x^2} \right) dx \\
 &= 7 \ln|x - 3| - \ln|x| + \frac{3}{x} + C.
 \end{aligned}$$

- [4] 3. First we write

$$g(x) = - \int_5^{x^3} \frac{\sin(t)}{t^2} dt.$$

Now we can apply the Fundamental Theorem of Calculus and the Chain Rule:

$$\begin{aligned}
 g'(x) &= - \frac{\sin(x^3)}{(x^3)^2} \cdot [x^3]' \\
 &= - \frac{\sin(x^3)}{x^6} \cdot 3x^2 \\
 &= - \frac{3 \sin(x^3)}{x^4}.
 \end{aligned}$$

- [9] 4. (a) The sketch of R is given in Figure 1. Note that the two curves intersect when

$$2x = 5 - 3x$$

$$5x = 5$$

$$x = 1.$$

By substitution back into either function, we see that this is the point $(1, 2)$.

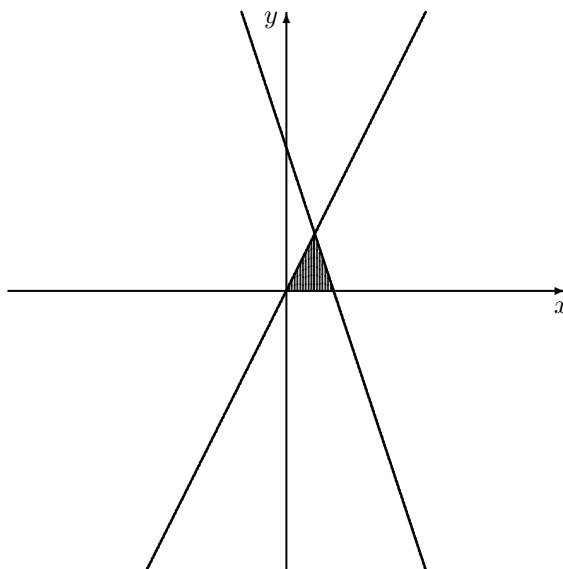


Figure 1: Question 4(a)

- (b) The region is not vertically simple, but it can be split into vertically simple regions on the intervals $[0, 1]$ and $[1, \frac{5}{3}]$. From the graph, we can see that the curve $y = 0$ is always the bottom boundary curve, while $y = 2x$ is the top boundary curve on $[0, 1]$ and $y = 5 - 3x$ is the top boundary curve on $[1, \frac{5}{3}]$. Thus

$$\begin{aligned} A &= \int_0^1 (2x - 0) dx + \int_1^{\frac{5}{3}} [(5 - 3x) - 0] dx \\ &= \int_0^1 2x dx + \int_1^{\frac{5}{3}} (5 - 3x) dx. \end{aligned}$$

- (c) The region is horizontally simple. The function $y = 2x$ can be written $x = \frac{1}{2}y$ (the lefthand boundary curve) while $y = 5 - 3x$ becomes $3x = 5 - y$ and so $x = \frac{5}{3} - \frac{1}{3}y$ (the righthand boundary curve). Hence

$$\begin{aligned} A &= \int_0^2 \left[\left(\frac{5}{3} - \frac{1}{3}y \right) - \frac{1}{2}y \right] dy \\ &= \int_0^2 \left(\frac{5}{3} - \frac{5}{6}y \right) dy. \end{aligned}$$